we could substitute *reflected abstraction* for *reflective abstraction* in Cifarelli and Sevim's original claim, it would agree with my analysis. Hence this is partially a matter of aligning terminology.

« 5 » First, let us review Piaget's characterization of reflective abstraction in general:

> "[Reflective] abstraction ranges over the coordinations of the subject's actions. These coordinations – and the reflecting process itself – may remain unconscious, or may give rise to consciousness…[It] involves two inseparable aspects. On the one hand, there is *projection* (as though by a reflective surface) onto a higher plane of what is drawn from the lower plane (for instance, from the plane of action to the plane of representation). On the other hand, there is *reflection,* a mental act of reconstructing and reorganizing on the higher plane what has been transferred by projection from the lower one." (Piaget 2001: 303)

« 6 » Utilizing this characterization, an analysis of Marie's activity reveals a series of reflective abstractions, with different levels of projection. That is, the "higher plane" (or elsewhere in Piaget, the "next developmental level") of projection is very gradually getting higher, i.e., Marie has been able to put an increasing distance between herself and her solution activities (cf. Piaget 2001: 306). Here I should note that my use of the term *projection* refers to the first part of the process of reflective abstraction discussed above. This is in contrast to the "reconstructing and reorganizing" of what has been projected, which Robert Campbell translated as *reflection.* This does not fit exactly with Cifarelli and Sevim's use of *reflection* as defined in §22. Their use of *reflection* seems to cover both parts of reflecting abstraction, as evidenced by their definition, which implies projection, and their subsequent use of the term in their categories of reflection, which implies reflection in my sense of the word. While I do not find their broader use of the word problematic in the context of their article, the distinction will be important in the present discussion.

« 7 » In fact, Cifarelli and Sevim's Levels of Reflection in Table 3 correspond well with levels of projection that Piaget (2001: 304) outlines. In Task 2, Marie could be interpreted to *recognize* some of the same quantitative elements as in Task 1, even before she solves the problem. This is equivalent to a first-level projection. She then constructs the new quantitative relationships in Task 2 and is able to *compare* that to Task 1 in order to identify potentially extraneous information. This is similar to a third-level project in Piaget's work and presupposes that at some point (either in Task 1 or Task 2), Marie has attentionally bounded the quantitative relationship from Task 1 as a *unified whole* that can be compared to the relationships in Task 2. This would correspond to a second-level projection. Cifarelli and Sevim's account of how Marie moves from recognition to a nascent anticipation of the results of her actions and of her ability to run through her actions using mental representations both serve as a nice elaboration of what comes between Piaget's third and fourth levels of projections. Finally, when Marie is able to run through the solution activity and evaluate the usefulness of the results in Task 9 (and arguably Task 4), Marie is at the fourth level of projection, which corresponds to *reflected abstraction* (cf. Campbell 2001).

« 8 » Therefore, I find that Cifarelli and Sevim's analysis of Marie's mathematical activity, particularly with a more psychological interpretation of *action*, provides an excellent illustration and elaboration of levels of reflective abstraction. Most of the other documentations I know of on reflective abstraction, outside of Cifarelli and Sevim's work, focus on the construction of new kinds of quantities or quantitative relationships. Marie's use of terms such as *relative heights*, along with her relative comfort in working with diagrams and equations in representing quantitative relationships, implies that Marie is not constructing a completely new type of mathematical structure. However, as opposed to decreasing the value of Cifarelli and Sevim's example, I think this increases its value in terms of implications for instruction.

« 9 » The more far-reaching discussion of objectification of actions that one finds in APOS theory (Asiala et al. 1996) or implicitly in the work of numerous researchers such as Leslie Steffe and John Olive's work on fraction learning (2010) has implications for broader curricular goals and choices. However, the changes they would hope to engender in students' ways of operating take months or even years to occur. In the meantime, it is the micro-scale reflective abstractions that we can hope to engender on a daily basis as educators. In §52, the authors argue that the importance of the general approach of encouraging problem solving and reflection are an implication of their research. I would agree, and further say that this may be the important implication of constructivist research in general on a daily scale.

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Re-presentations and Conceptual Structures of What?

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> Upshot • Education researchers often explain student activity in terms of general thinking and learning processes, including those identified by Cifarelli and Sevim. In this commentary, I refocus Cifarelli and Sevim's discussion in order to hypothesize the organization of mental actions that comprise and support those learning processes.

« 1 » In their target article, Victor Cifarelli and Volkan Sevim provide an example of a student (Marie) developing abstract levels of solution activity over the course of a task-based interview. Describing Marie's activity in solving Task 1, the authors claim, "[Marie] developed an action pattern from her solution activity while solving Task 1, which served as a conceptual structure that enabled her to interpret new tasks as similar, and thus assimilate new situations to her current structure" (§28). The authors then conclude that Marie's conceptual structure became more abstract as she reconstructed this conceptual structure in subsequent tasks, eventually enabling her to anticipate or re-present her solution activity independent of contextual details and without carrying out the activity.

« 2 » Marie's progress during the sequence of tasks leaves the following questions: What organization of mental actions[1] composed her conceptual structure and how did the nature of these mental actions support her development of an increasingly abstract conceptual structure? In this commentary, I provide one perspective on the mental actions driving Marie's actions and conclude with thoughts on relationships between the posed perspective, Cifarelli and Sevim's work, and broader work in mathematics education.

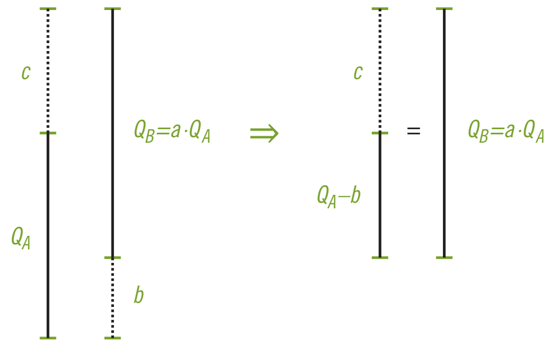

Figure 1: A hypothetical quantitative structure.

Processes as mental actions

« 3 » Cifarelli and Sevim attribute Marie's development to processes including re-presentation, recognition, reflection, and abstraction, each of which von Glasersfeld considered critical to explaining the nature of knowing and learning. Von Glasersfeld also considered each of these processes as sensitive to the particular mental actions and patterns that comprise the processes. For instance, von Glasersfeld described recognition as requiring, "the attentional selecting, grouping, and coordinating of sensory material that fits the composition program of the item to be recognized" (Glasersfeld 1991: 5). That is, recognition is not a process in and of itself; rather, recognition entails particular mental actions and assimilation based on previous experience. Likewise, von Glasersfeld considered re-presentation to involve the enactment and coordination of particular mental actions, but that re-presentation differs from recognition in that re-presentation involves the enactment and coordinating of mental actions without the available sensory material that first gave rise to the mental actions (Glasersfeld 1991). It is for this reason that von Glasersfeld considered re-presentation to involve a higher level of abstraction than recognition.

« 4 » Compatible with and informing von Glasersfeld's treatment of the aforementioned processes, Jean Piaget characterized thinking and learning in terms of dynamic processes and organizations of mental actions (Piaget 1977). For instance, Piaget described different kinds of abstraction based on the locus of the entailed schemes of mental actions. Two kinds of abstraction are pseudo-empirical abstractions and reflective abstractions. Whereas pseudo-empirical abstractions are based on the results of activity (both in the physical and the mental sense), reflective abstractions are based in the coordination and re-presentation of the activity itself. Because of these differences, Piaget necessarily provided models of children's thinking at the level of mental actions in order to distinguish between kinds of abstraction when describing children's thinking and learning.

« 5 » Over the past few decades, numerous mathematics educators have conducted research in ways that are attentive to Piaget and von Glasersfeld's sensitivity to mental actions. Leslie Steffe's[2] body of work on students' fractional and multiplicative reasoning provides an apropos example (see Steffe & Olive 2010 for a comprehensive overview). Although Steffe's work includes a focus on students' abstraction with respect to their fractional knowledge, I interpret his primary objective being the characterization of the organization of mental actions entailed at each stage or level of abstraction (e.g., various levels of units coordination). It is from his descriptions at the level of mental actions and coordination that he has been able to hypothesize abstractions that take place during students' units coordination development. In other words, his descriptions of mental actions and coordination enable him to provide models of what it is that students re-present and abstract.

Quantitative structures and abstraction

« 6 » A line of research that I find relevant to Cifarelli and Sevim's work, as well as including a focus on mental actions critical to abstraction, is that which characterizes students' quantitative reasoning (Thompson 1993). Work in this area (Castillo-Garsow 2012; Ellis 2007; Johnson 2012) characterizes students' thinking and learning in terms of their constructing and abstracting measurable attributes (e.g., quantities) and relationships between these attributes. As an example, Patrick Thompson (1994) characterized relationships between rate and the abstraction of constructing and coordinating proportionally accumulating quantities. In short, Thompson argued that a student reflectively abstracted notions of rate through repeatedly (re)constructing and coordinating quantities.

1 | For the purpose of this commentary, I use mental actions to refer to both mental operations and mental actions. Mental operations are those mental actions that are reversible.

2 | Steffe has worked with numerous colleagues in this area, including Paul Cobb, Amy Hackenberg, Anderson Norton, John Olive, Erik Tillema, Patrick Thompson, and Catherine Ulrich, to name a few. For brevity's sake, I only refer to Steffe.

« 7 » Returning to Marie, a possible explanation for her construction of an increasingly abstract conceptual structure is that she had repeatedly constructed and coordinated particular quantities and relationships that resulted in an increasingly abstract *quantitative structure* (Thompson 1994). Figure 1 provides a hypothetical model of this quantitative structure. The quantitative structure (Figure 1) is composed of two lengths, Q_A and Q_B, beginning and ending at (potentially) different reference points, thus forming the lengths of b and c to account for the amount by which Q_A and Q_B exceed Q_B and Q_A, respectively. Also, Q_B is known to be equivalent to a times the length Q_A. From here, it can be deduced that Q_B is equivalent to the sum of c and the difference between Q_A and b (e.g., Figure 1, right, and Marie's equation of $x - 15 + 50 = 2x$ in Task 4).

« 8 » In characterizing Marie's solution activity on Task 1, Cifarelli and Sevim describe that she constructed a diagram that "aided her understanding of the situation" (§27). One way to explain how the diagram "aided her understanding of the situation" is that Marie constituted her diagram in terms of quantities and relationships between quantities that she then symbolized using equations and variables, whereas her initial approach to the problem was entirely symbolic and devoid of imagined quantities and relationships. As Marie then progressed to subsequent tasks, during which she seemed to spend more time focusing on her image of the situation and its quantities (§28), she was able to recognize that she was constructing similar quantitative structures. That is, reconstructing, recognizing, and re-presenting quantitative structures enabled her to construct a sense of invariance or similarity across the tasks, eventually leading to her anticipating the implications of this structure (e.g., solutions, calculations, and the necessity of particular information and values).

« 9 » Echoing von Glasersfeld's approach to recognition and re-presentation, I find it important to emphasize that the assimilation of a situation to a quantitative structure is not a passive absorption of information (Thompson 2013). Rather, assimilation to a quantitative structure involves constituting the situation such that it entails that quantitative structure and its implications, with such activity looking different at various levels of abstraction (e.g., the extent that the assimilation entails anticipatory actions). Hence, each time Marie came to a new situation, she had to conceive the situation so that it entailed a quantitative structure, and it was through this repeated process that she constructed an increasingly abstract and anticipatory structure.

Coordinating foci

« 10 » Research on students' mathematical thinking and learning encompasses a number of foci ranging from specifying students' mental actions to characterizing general activity and learning processes. Regardless of the primary foci of research, I contend that inquiry into student thinking and learning must be sensitive to the particular mental actions at play. If not, we can end up taking structures and processes for granted as opposed to being inherently tied to the organization of mental actions that constitute the structures and processes (see Dawkins 2012 for a discussion of this phenomenon).

« 11 » An important area moving forward is the determination of systematic ways to coordinate research foci in order to construct more extensive models of student thinking and learning. For instance, there is a significant body of literature on students' problem solving processes, with several researchers providing frameworks by which to categorize students' activity (e.g., Carlson & Bloom 2005; Pólya 1973). An open question thus becomes: Can we create relationships between problem solving frameworks and those constructs that are more fine-grained in terms of mental actions and processes of mathematical thought? For instance, there seems to be a natural connection between Carlson and Bloom's conjecture-imagine-evaluate cycle, students' quantitative reasoning, and their abstraction of conceptual structures; a student that has abstracted particular quantitative structures so that these structures are anticipatory might be supported in mentally running through conjecture-imagine-evaluate cycles. As another example, it seems intuitive that students' cycling back or checking activity involves processes of re-presentation and reflection. What mental actions might support such processes as being more productive than not in the moment of problem solving? The answers to these questions will no doubt further our understanding of student thinking and learning.

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Mathematical Modeling and the Nature of Problem Solving

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> **Upshot** • Problem solving is an enormous field of study, where so-called "problems" can end up having very little in common. One of the least studied categories of problems is open-ended mathematical modeling research. Cifarelli and Sevim's framework – although not developed for this purpose – may be a useful lens for studying the development of mathematical modelers and researchers in applied mathematics.

« 1 » Victor Cifarelli and Volkan Sevim's target article bears superficial similarity to some of my own work in studying student's mathematical development (Castillo-Garsow 2010, 2012, 2013; Castillo-Garsow, Johnson, & Moore 2013). We both study the mathematical development of individual students engaging in structured series of mathematical tasks; however, applications differ. My work focuses on the development of particular mathematical tools for modeling, while the authors propose a framework (§46, Table 3) for how a student might expand the scope of a problem to see other