Coming to Our Senses: From Constructivism to Democratization of Math Education

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Motivation: Paralleling my own transformation from a Platonist to a radical constructivist, mathematics education has been experiencing for more than a decade a movement that started in theoretical foundations mostly originating in von Glasersfeld's work, and then reached professional organizations, which have been leading extensive efforts to reform school mathematics according to constructivist principles. However, the theories espoused by the researchers are, as yet, too abstract to lend themselves readily to implementation in the classroom. ***Purpose:*** I define a shared experiential language (SEL) for the constructivist teacher to *embody* in order to transform her practice congruently according to constructivist principles. While SEL is comprised of Neuro-Linguistic Programming (NLP) subjective experience distinctions, what "makes it tick" is the *constructivist epistemology* with its insight that for consistent understanding to happen, new knowledge has to attach to prior experiences in a process of co-construction. Throughout the paper, I elaborate and validate this insight by numerous examples. ***Practical implications:*** Utilizing SEL allows understanding of mathematics to be rooted in each student's individual sensory experiences, thus shifting the responsibility for success in mathematics from the students back to those who guide them in co-constructing knowledge. This, in turn, should allow everybody access to understanding and so it should no longer be socially acceptable to fail in mathematics. ***Key words:*** Radical constructivism, math education, Neuro-Linguistic Programming, sensory experience, behavioral cues, democratization.

The child cannot conceive of tasks, the way to solve them and the solutions in terms other than those that are available at the particular moment in his or her conceptual development. The child must make meaning of the task and try to construct a solution by using material she already has. That material cannot be anything but the conceptual building blocks and operations that the child has assembled in his or her own prior experience. — Glasersfeld (1987, p. 12)

Introduction

Having been trained in the Platonism of traditional mathematics, my first "Learning III" experience that Bateson defined as one in which "there is a profound reorganization of character" (Bateson 1972, p. 301) occurred in the late 80s when I started studying Neuro-Linguistic Programming (NLP). NLP is a set of models of subjective experience created for the purpose of making explicit and emulating in oneself and others strategies of excellence (Dilts et al. 1980). Its primary tenet (formulated originally by A. Korzybski) is "The map is not the territory," which, in the words of Watzlawick (1984), means that "the name is not what it names; an interpretation of reality is only an interpretation and not reality itself. Only a schizophrenic eats the menu instead of the foods listed on the menu" (p. 215). I embraced this tenet, and as a consequence a shift in my world view occurred that turned my life around: I moved from the modernist's belief in an objective reality accessible by reason and observation to the postmodernist's belief in subjectivity (Pasztor & Slater 2000).

Having grown up in a communist country, Watzlawick's (1984) words struck a chord with me: any system that denies that it operates on a map of reality, rather than on reality itself, will not only be unable to recognize and adjust to changes in its perception of reality, but will also be unable to tolerate any other representation of reality. I have had first hand experience of examples that "go from the ridiculous to the gruesome" of a totalitarian regime's "paradoxical, recursive logic" that typically characterizes paranoia: "It is inherent to the concept of paranoia that it rests on a fundamental assumption that is held to be absolutely true. Because this fundamental assumption is axiomatic, it cannot and need not demonstrate its own veracity. Strict logical deductions are then made from this fundamental premise and create a reality in which any failures and inconsistencies of the system are attributed to the deduction, but never to the original premise itself" (ibid., pp. 223–224). Whoever criticizes the premises of the system is therefore declared to be an enemy and will not be tolerated.

Ten years later, my then therapist and now co-author and friend, Mary Hale-Haniff, introduced me to constructivist therapies. What a shake-up I had when I read in Lynn Hoffman's (1990) paper an account of Heinz von Foerster criticizing NLP's tenet, "The map is not the territory," and confronting it with his own view that "*The map* is *the territory*"! Once again, I embarked on a "Learning III" experience, and it all fell into place when I read von Glasersfeld's (1984) introduction to radical constructivism. It clicked. It *fit* perfectly with most aspects of my life – some conscious, some unconscious. It *fit* with my dissatisfaction with the hierarchical teacher–student, physician–patient, therapist–client and other similar relationships, and my deep distrust of statistics and other quantitative research methodologies. I came to understand that NLP's epistemology was incongruent with its overall intents. Its map-territory distinction presupposed the existence of a reality that preexists the observer and from which information is filtered onto our individual maps (Hale-Haniff 2004). NLP models were designed to "force" the client to change limitations in her map. Thus, the therapist–client relationship once again became a hierarchical and coercive one.

I was suddenly able to observe that numerous fields such as education, science, psychotherapy, linguistics, organizational studies, etc., were undergoing a paradigm shift from positivism to constructivism, to a world view in which adherence to authority and external control is replaced by reliance and trust in subjective experience. This, in turn, would necessarily lead to a democratization of the respective field, since knowledge or expertise is not the privilege of a small "talented" elite, but can be constructed by each person according to their previous experience.

Hardest to understand in the shift to radical constructivism – even to the theorist – is the distinction between its tenets and statements such as 'there exists an external reality, but we do not have direct, unmediated access to it' or 'there exists no independent reality.' In my contribution, I will illustrate the practical impact of such a distinction on mathematics education. In particular, I will focus on the democratization of mathematics (cf. Pasztor 2004a) – the shift away from the "transmission" model of education towards a theory of knowledge and a new methodology, in which the process of understanding or coming to know is a matter of constructing, from elements available in the student's own experience, conceptual structures that lead to "a viable path of action, a viable solution to an experiential problem, or a viable interpretation of a piece of language," and "there is never any reason to believe that this construction is the only one possible" (Glasersfeld 1987, p. 10).

Von Glasersfeld's writings are among the very few academic ones that have deeply affected my personal life as well (as if there was a non-personal life …) Sometimes, when I ask my husband to, say, put the garbage out and he fails to do so and later I question him about it, he may reply, "But you didn't tell me to do so." In such a case, I respond, "You cannot say that I didn't *tell* you, the only thing you can say is that you didn't *hear* me tell you." Thus, von Glasersfeld entered our marital life.

The traditional approach to mathematics education

The traditional, positivist approach to instruction has been referred to as "the Age of the Sage on the Stage" (Davis & Maher 1997, p. 93), due to its "transmission" model of teaching, where teaching means "getting knowledge into the heads" of the students (Glasersfeld 1987, p. 3), that is, *transmitting* knowledge from the teacher to the student. The underlying philosophy is that knowledge is out there, independent of the knower, ready to be discovered and be transferred into people's heads. It is "a commodity that can be communicated" (Glasersfeld 1987, p. 6). The *ontology* presupposed in this view is that there is one true reality out there, which exists independently of the observer. Furthermore, we have access to this reality, and we can fragment, study, predict and control it (Lincoln & Guba 1985; Hale-Haniff & Pasztor 1999).

However, as von Glasersfeld (1987, p. 4) points out, while trying to access reality, we have been caught in an age-old dilemma: If truth is defined as "the perfect match, the flawless representation" of reality, *who is to judge "the perfect match with reality"?*

To answer this question, Western philosophy has taken a route in which, given the right tools, pure reason is believed to be able to transcend all social and cultural constraints and the confines of the human body, including those of perception and emotion. Mathematical reasoning has been seen as the purest example of reason: "purely abstract, transcendental, culture-free, unemotional, universal, decontextualized, disembodied, and hence formal" (Lakoff & Nuñez 1997, p. 22; for more "fine-tuned" criticism cf. Lakatos 1976). The traditional scientist, mathematician, or, in general, researcher, is out to find objective truth. In doing so, she is trained to be value-neutral in order to be able to objectively judge "the perfect match" with reality. She is a "cool, detached, solitary genius, the one who has the answers that others don't have, as if the truth could be owned" (Pert 1997, p. 315).

In practice, however, there is a direct "relationship between claims to truth and the distribution of power in society" (Gergen 1991, p. 95). Those at the top of the educational system hierarchy are the "objective" experts of knowledge; they determine teaching goals and criteria of assessment. Accordingly, the traditional teacher–student relationship is a hierarchical, authoritarian relationship.

The constructivist view of knowledge and its implications for mathematics education

In contrast to positivist philosophy, constructivist philosophies have adopted a concept of knowledge that is *not* based on any belief in an accessible objective reality. In the radical constructivist view, knowing is not matching reality, but rather finding a *fit* with observations. Constructivist knowledge "is knowledge that human reason derives from experience. It does not represent a picture of the 'real' world but provides structure and organization to experience. As such it has an all-important function: It enables us to solve experiential problems" (Glasersfeld 1987, p. 5). With this theory of knowledge, the experiencing human turns "from an explorer who is condemned to seek 'structural properties' of an inaccessible reality … into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives" (ibid.).

Now, let us look at the two views that are so often confused with the tenets of radical constructivism (Pasztor 2004a): 1. there exists a mind-independent reality (MIR), albeit only indirectly accessible, and 2. there exists no MIR. The first view is close to the positivist ontology, except now we do not have the possibility of a "perfect match," but only that of a mediated match. Still, *who is going to judge the "better" match*?

A constructivist view is inconsistent with *both* of these ontological views. As von Glasersfeld (2004a, [2]) states, the constructivist holds "that all coordination and, therefore, all structure is of the organism's own making," and therefore he has no way of knowing anything about the ontological reality of these constructs. In fact, he has no way of knowing anything about an MIR. Furthermore, as soon as we posit the existence or nonexistence of an MIR, we have caused a split between the knower and the known. The one who knows whether an MIR exists or does not exist becomes the expert, the authority. In the constructivist view, a third person has no way of knowing anything about my or your own experience. As von Glasersfeld (2004b, [4]) says, "'someone else' is always my construction." The only expert of your experience is you. This view, as I will show, can make a tremendous difference in math education.

For more than a decade now, mathematics education in the US has been experiencing a top-down reform movement that started with the theoretical foundations of mathematics education that mostly originated in von Glasersfeld's work, and then moved to the professional organizations, which then started and have since been leading extensive efforts to reform school mathematics according to constructivist principles (NCTM 2000). So far, however, the reform has been moving only very slowly into the mathematics classroom practices. Besides complex political reasons (Alacaci & Pasztor 2002), one of the reasons for this is that the theories espoused by the researchers to implement constructivist principles are, as yet, too abstract to readily lend themselves to implementation. One of the goals of my own research efforts in math education has been to help translate the language of these theories into the experiential language of students.

Abstract mathematical concepts are *metaphorical* and are built from people's sensory experiences (Lakoff & Nuñez 1997; Lakoff & Johnson 1999). The constructivist teacher's role is to make sure that they *fit* the students' *individual experience*. Frustration and confusion ensue if the teacher's metaphorical mapping is rooted in an a-priori construction, rather than in the student's own experience. English (1997) provides a very good example of what happens in such a case. It concerns the use of a line metaphor to represent our number system, whereby numbers are considered as points on a line. The "number line" is used to convey the notion of positive and negative number, and to visualize relationships between numbers. It turns out that students frequently have difficulty in abstracting mathematical ideas that are linked to the number line (Dufour-Janvier, Bednarz & Belanger 1987, quoted in English 1997, p. 8). "There is a tendency for students to see the number line as a series of 'stepping stones,' with each step conceived of as a rock with a hole between each two successive rocks. This may explain why so many students say that there are no numbers, or at the most, one, between two whole numbers."

While students are often able to reorganize their experience in a way that makes it *fit* the constraints of the problem at hand, often times the teacher needs *to provide for the students* "precisely those experiences that will be most useful for further development or revision of the mental structures that are being built" (Davis & Maher 1997, p. 94). This idea is wonderfully demonstrated by Machtinger (1965) (quoted in Davis & Maher, 1997, pp. 94–95) who taught kindergarten kids to conjecture and prove several theorems about numbers, including even + even = even, even + odd = odd, and odd + odd = even. She did so by defining a number *n* as "even" if a group of *n* children could be organized into pairs for walking along the corridor and as "odd" if such a group had one child left over when organized into pairs. Since walking along the corridor in pairs was a daily experience for the kids, learning the new information became a matter of just expanding or reorganizing their existing knowledge.

But this is *not* always possible. In particular it is not possible when the teacher uses incompatible metaphors to explain mathematical ideas. I was shocked and saddened by the great regret with which the 86-year-old Carl Jung remembered in his 1962 memoirs the terror that he experienced in math classes. While his teacher gave the impression that algebra was very natural, the young Jung failed to understand what numbers actually were. He knew they were not flowers, nor animals, nor fossils – they were nothing he could imagine. They were just amounts that resulted from counting. To his greatest confusion, these amounts were replaced by letters the meaning of which was a sound. His teacher tried hard to explain the purpose of this strange operation of replacing understandable amounts by sounds, but to no avail. This, what seemed to Jung to be a random expression of numbers through sounds such as "a," "b," "c," or "x," did not explain anything about the nature of numbers. His frustration peaked with the axiom, "if a = b and b = c, then a = c," since by definition it was clear that "a" denoted something different from "b," and so could not be equaled with "b," let alone with "c." He was outraged. An equality could be "a = a," but "a = b" was a lie and deceit. His intellectual morality resisted such incongruities that blocked his access to the understanding of mathematics. To his old age Jung had the uncorrectable feeling that if he could have accepted the possibility of "a = b," that is, of "sun = moon, dog = cat, etc.," then mathematics would have infinitely absorbed him. Instead, he came to doubt the morality of mathematics for his entire life. Like so many others, he came to doubt his own self-worth, which, back then, prevented him from asking questions in class (Jung 1962).

In practice, "[f]or too many people, mathematics stopped making sense somewhere along the way. Either slowly or dramatically, they gave up on the field as hopelessly baffling and difficult, and they grew up to be adults who – confident that others share their experience – nonchalantly announce, 'Math was just not for me' or 'I was never good at it.'" (Askey 1999). Ruth McNeill shares her story of how she came to quit math: "What did me in was the idea that a negative number times a negative number comes out to a positive number. This seemed (and still seems) inherently unlikely – counterintuitive, as mathematicians say. I … could not overcome my strong sense that multiplying intensifies something, and thus two negative numbers multiplied together should properly produce a *very* negative result" (McNeill 1988, quoted in Askey 1999).

Most mathematical concepts being metaphorical and understanding a metaphor meaning successfully mapping concepts from our individual experience onto new domains, teaching the metaphorical structure of mathematics becomes indispensable. It shifts the definition of "mathematical understanding" from a goal that only a few "talented" or "gifted" people can reach, to a process rooted in *all* people's individual experience.

Is 2 + 2 still 4?

If objectivity of mathematics is just a myth, what happens to basic facts such as "2 + 2 = 4?" Are we denying them? The question is very nicely answered in a dialogue between von Foerster and von Glaserfeld in their (1999) book. The following is an excerpt from the book (translated from German by myself).

von Glasersfeld: "Mathematics is of course a free invention, but very often this is misunderstood, because people say, 'Well, if it is freely invented, why is 2 × 2 always 4?' … The free invention of course doesn't mean that once you have assumed certain rules, you may intentionally break these rules. It is just like in chess, where you assume that the chess figures move in a certain way. The situations that you then construct, and the moves that are then possible, arise as consequences of applying the accepted rules. As I see it, this is the same in math. There one creates certain rules, and the first rules concern numbers. Counting rests on more complicated rules than most people are aware of. They can count, but are not always clear about everything they do while counting. … To count, you must first have the concept of unit. Then you must perceive units, that is, you must be able to construct them according to your perception. You have to be able see them, or show them, or push them on a table, or shift them on a rod on the abacus. And with each unit that you shift, you have to utter one of the numerals of a fixed sequence of numerals. You must not alter the sequence. If you follow these rules then it is no magic that 2 + 2 is always 4. You could only get a different result if you suddenly started counting, '1, 2, 7, 6' instead of the normal order, thus breaking an accepted rule. In that case 2 + 2 would be 6."

von Foerster: "That would be like playing chess and moving the threatened king two squares instead of one. Then you would be stepping out of the game."

von Glasersfeld: "Yes – and if my opponent explained why this is so, then I would discover that I broke a rule. This also shows that it is the rules that determine when my king is in check-mate. We don't invent this during the game …."

von Foerster: "In mathematics this is of course the same – here the rules imply a variety of things that one could not easily have predicted."

von Glasersfeld: "Piaget has this nice example where a child first finds out that it makes no difference whether he counts eight marbles placed in a circle clockwise or counterclockwise. It always amounts to 8. And Piaget puts it very nicely that this 8 is not a perceived fact, but the result of rule-based actions. As long as we perform these actions according to the rules, we come to the result determined by these rules. And with the action of counting the directions plays no role, but according to the rules, we may count each unit only once. This is the number constancy" (Foerster & Glasersfeld 1999, pp. 133–134).

So, while mathematics is a human construction, it is not an arbitrary creation. It is "not a mere historically contingent social construction. What makes mathematics non-arbitrary is that it uses the basic conceptual mechanisms of the embodied human mind… Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history" (Lakoff & Nuñez 2000, p. 9).

Operative learning and learning states

In constructivism, the meaning of learning has shifted from the student's "correct" replication of what the teacher does to "the student's *conscious understanding* of what he or she is doing and why it is being done" (Glasersfeld 1987, p. 12). "Mathematical knowledge cannot be reduced to a stock of retrievable 'facts' but concerns the ability to compute new results. To use Piaget's terms, it is *operative* rather than *figurative*. It is the product of reflection – and whereas reflection as such is not observable, its product *may* be inferred from observable responses" (Glasersfeld 1987, p. 10). Operative knowledge is constructive. "It is not the particular response that matters but the way in which it was arrived at" (Glasersfeld 1987, p. 11).

But how is the student to attain such operative knowledge in mathematics, when the "structure of mathematical concepts is still largely obscure" (Glasersfeld 1987, p. 13)? Most definitions in mathematics are *formal* rather than *conceptual*. In mathematics, definitions "merely substitute other signs or symbols for the definiendum. Rarely, if ever, is there a hint, let alone an indication, of what one must *do* in order to build up the conceptual structures that are to be associated with the symbols. Yet, that is of course what a student has to find out if he or she is to acquire a new concept" (Glasersfeld 1987, p. 14).

To illustrate this point, let us look at an example. While talking about my research to J, a doctoral student in Computer Science in his mid thirties, I asked him to solve a word problem. "Word problem? I *hate* word problems!" was J's response even before he knew what the word problem was. The word problem was this: "Joey has a new puppy. His sister, Jenna, has a big dog. Jenna's dog weighs eight times as much as the puppy. Both pets together weigh 54 pounds. How much does Joey's puppy weigh?" J listened to the problem, and then asked me to repeat it. As I did so, J made the following notes, turning his back to me:

puppy: x
big dog: $8x$
$x + 8x = 54$
$9x = 54$

Then he stopped and said he didn't know his multiplication table. "So anyway, what is the answer?" I asked. J blushed and became restless. "What do you mean?" he asked. I replied, "Well, what was the question?" After Jeff repeated the problem's question, I asked again, "So, how much does the puppy weigh?" Again, J didn't answer but became instead more and more insecure. "Why, did I do something wrong? I must have screwed up somewhere." "No," I replied. "All I have in mind is *how* do you get that x?"

J was so fixed on getting the exact number as a result, that it never occurred to him to say something like "The puppy weighs 54 divided by 9, whatever that is." Instead, he questioned his whole approach thinking he had "screwed up somewhere." I asked him why he hated word problems. He replied, "Because they make me feel stupid." How? I inquired. "Well, if I don't get an immediate answer, I feel stupid. It is stuff I should know. It is expected of me." Jeff went on to talk about the time when he came to hate word problems. He never understood what the teacher did in class – he failed to see any pattern in these word problems. The teacher had them solve word problems either under time pressure or at the board, in front of the entire class. He felt threatened and never actually got over it.

There is a general agreement across the constructivist research in mathematics education that for consistent understanding to happen, new knowledge has to attach to students' prior experiences. But just what *kind* of prior experiences? Which ones are optimal for new learning? How can a teacher behave in a way as to resurrect those experiences? What are resource states of learning? How are attentional units of those states configured? How can a teacher know when she is eliciting an un-useful experience? Even though people's subjective experiences are private, can students and teachers come to share a language of experience? How?

Making *sense* of math – literally!

These and similar questions have guided my work in the last two decades, helping me set research goals such as exploring the relationship between mathematical knowledge and the subjective experience it gets attached to in the process we call understanding.

While holding a constructivist epistemology, I have been able to facilitate successful mathematics understanding in my students by using a shared experiential language (SEL) that allows a direct, two-way communication between the teachers and students. SEL is based on NLP models and comprises categories of subjective experience such as sensory (see-hear-feel) modalities, submodalities, sensory strategies, and behavioral cues, as well as ways for the teachers to separate student's meanings from their own (Hale-Haniff & Pasztor 1999; Hale-Haniff 2004; Pasztor 2004b).

Sensory modalities: The see/hear/feel building blocks of our experience

According to Damasio (1994), at each moment in time our subjective experience is manifested in what he calls an "image": a *visual image*, that is, an internal picture; an *auditory image*, that is, sounds – discrete or analog; a *kinesthetic image*, that is, a feeling or an internal smell or taste; or a combination of these. For example, while J's representation of "even number" is manifested in a fuzzy visual image of the number two, accompanied by "a feeling of 2ness," and my own representation is a sharp visual image of "2*n*," written in white on a blackboard and situated right in front of me, my friend Mary represents "even number" by hearing the actual definition of "even number."

Many people argue that they do not think in images, but rather in words or abstract symbols. But "most of the words we use in our inner speech, before speaking or writing a sentence, exist as auditory or visual images in our consciousness. If they did not become images, however fleetingly, they would not be anything we could know" (Damasio 1994, p. 106).

Damasio (1994) goes as far as to require as an essential condition for having a mind the ability to form internal (visual, auditory, kinesthetic) images, and to order them in the process we call thought. His view is that "having a mind means that an organism forms neural representations which can become images, be manipulated in a process called thought, and eventually influence behavior by helping predict the future, plan accordingly, and choose the next action" (p. 90).

Sensory images are often referred to as "mental representations" – a term that, as von Glasersfeld (1987) explains, can be quite misleading: "In the constructivist view, 'concepts,' 'mental representation,' 'memories,' 'images,' and so on, must not be thought of as static but always as *dynamic*; that is to say, they are not conceived as postcards that can be retrieved from some file, but rather as relatively self-contained programs or production routines that can be called up and run (cf. Damasio's 1994 dispositional representations). Conceptions, then, are produced internally. They are replayed, shelved, or discarded according to their usefulness and applicability in experiential contexts. The more often they turn out to be viable, the more solid and reliable they seem. But no amount of usefulness or reliability can alter their internal, conceptual origin. They are not replicas of external originals, simply because no cognitive organism can have access to 'things-in-themselves' and thus there are no models to be copied" (p. 219).

How constructivism honors other ways of knowing and communicating

Positivist methodology privileges auditory-verbal communication, often to the exclusion of other modalities. Thus we teach the verbally oriented conscious mind, and often ignore visual and kinesthetic aspects of experience. However, if we intend to communicate in a holistic manner *engaging all of our senses*, we need to also honor other ways of knowing. "For the constructivist teacher – much like the psychoanalyst – 'telling' is usually not an effective tool. In this role, the teacher is much less a lecturer, and much more of a coach (as in learning tennis, or in learning to play the piano). A recent slogan describes this by saying 'the Sage on the Stage has been replaced by the Guide on the Side.' It is the *student* who is doing the work of building or revising [… his or her] personal representations. The student builds up the ideas in his or her own head, and the teacher has at best a limited role in shaping the student's personal mental representations. The experiences that the teacher provides are grist to the mill, but the student is the miller" (Davis & Maher 1997, p. 94).

The holistic, constructivist view presupposes that the teacher should have the potential to attend to all aspects of sensory experience and communication *both* in herself and in the student's system. In addition to auditory-verbal aspects, visual and kinesthetic experience may also be privileged, with both unconscious (tacit) and conscious communication and perception considered. When teachers are (implicitly) trained to ignore communications related to intra-personal, emotional, and unconscious experience, we are imparting positivist principles. Most of us have been socialized largely according to positivist thinking, conceptualizing emotions as sudden and intense experiences that come and go at certain times; something that a sane or balanced person learns to keep under control so that rational thinking and control can prevail. On the other hand, the holistic, constructivist view depicts emotional experience as ongoing, simultaneous with and supportive of, the rest of experience.

Kinesthetic experience is ever-present (although not always consciously accessible) in form of "body images." "By dint of juxtaposition, body images give to other images a quality of goodness or badness, of pleasure or pain. I see feelings as having a truly privileged status… [F]eelings have a say on how the rest of the brain and cognition go about their business" (Damasio 1994, pp. 159–160).

It is important to note that experience that is kinesthetic to one person (e.g., the student) is accessible primarily visually to the other (e.g., the teacher). For example, as the student feels his or her face get hot, the teacher might notice him blush. Or, as the student feels a sense of pride welling up in him, the teacher might notice him taking a deep breath as he squares his shoulders. Thus learning to detect new categories of sensory experience in oneself and others involves enhancing perception of new categories of both kinesthetic and visual experience. Becoming more consciously aware of categories of sensory experience other than auditory-verbal, the teacher enhances her ability to accommodate to the students' experiences.

Submodalities: Refining the see/hear/ feel building blocks

Each sensory modality is designed to 'perceive' certain basic qualities called *submodalities*, of the experience it represents (Bandler & MacDonald 1988; Pasztor 1998; Hale-Haniff & Pasztor 1999). *Visual* submodalities refer to qualities such as: location in space, relative size, hues of color or black and white, presence or absence of movement, rhythm, degree of illumination, degree of clarity or focus, flat or three-dimensional; associated or dissociated (seeing oneself in the image, or viewing from a fully associated position). *Auditory* submodalities refer to qualities such as location, rhythm, relative pitch, relative volume, content: voice, music, noise. *Kinesthetic* submodalities include such qualities as: location of sensations, presence or absence of movement (and if moving, the physical locations of sequential sensations), the type of sensations: temperature, pressure, density, duration, moisture, pervasiveness of body area involved, sense of movement and acceleration, changes in direction and rotation.

Submodalities are distinctions that separate experiences from one another. As such, their significance comes to bear only when we contrast submodalities of images that represent different experiences. To illustrate this, let us look at the submodalities of different experiences of my husband, specifically at how different contexts are manifested in completely different sets of submodalities. My husband is an architect and he is quite proficient in geometry. First, here is what he reports regarding his experience of abstraction: "As part of a math problem involving triangles, an *abstract* triangle occurs first as a fuzzy shape without any material 'body.' It doesn't have a surface; not even a clear boundary. Its size is also changing between a couple of inches to one or two feet. It is quite far from my face and its distance is unspecific but it is still in the room. As a consequence, its shape, size, and location can easily be manipulated. As it is manipulated, such as made equilateral or rotated, these parameters change rapidly. The boundary becomes more defined, the size concrete, and the distance fixed. It still remains, however, a line-drawing without a body or surface. It is always a colorless figure, either gray or black and white." In contrast, for my husband imagining an emergency triangle on the road propped up behind a car "is a vivid picture with concrete shape, thickness, material, etc. It is red with white edges in fluorescent colors set against the gray asphalt background. I see it at a distance of 10 feet in life size, that is, the same size I would probably see it driving by and looking at it from this same distance. I feel some anxiety in my stomach as I probably connect this picture unconsciously with a car break-down or an accident."

Sensory strategies: sequences of see/hear/ feel blocks leading to a particular outcome

Our thought processes are organized in sequences of images that have become consolidated into functional units of behavior leading to a particular outcome and often executed below the threshold of consciousness. They are called *sensory strategies* (Dilts et al. 1980) Each image triggers another image or a sequence of images. For example, you hear X's name, this triggers your remembering X's face, close up, somewhat distorted, and pinkish red, which, in turn, triggers a negative feeling. Over time, each image or sequence of images comes to serve as a stimulus that automatically triggers other portions of the perceptual or recalled experience it represents. The creation of such triggers happens through learning and depends on various complex subjective, social, cultural and other factors. I will illustrate the idea of sensory strategy with a few examples from a pilot project I conducted in the academic year 1999–2000 with a class of fourth graders with the aim of teaching them SEL and through it, awareness of their mental processes while solving math problems.

Ramon chose the following problem to solve: *Which measure is the best estimate to describe the length of the salamander below (picture followed text)? Circle the best estimate: 3 inches 3miles 3 pounds.*

Here is what he reported: "What I did was picture a huge ruler in front of my face and I saw the numbers 1,2,3,4,5,… I looked at the picture [in the book] and compared it with 3 inch and it was right. Besides, pounds is weight and miles is larger than inch."

Kevin's strategy for implementing a pattern is also quite remarkable. I asked the class to multiply 1 × 1 (= 1), 11 × 11 (= 121), 111 × 111 (= 12321), and 1111 × 1111 (= 1234321). Then I asked them to continue the pattern. Kevin reported the following for calculating 11111 × 11111: "First I looked, then [knocking with his left hand on his head just above his left ear] I heard 'tap, tap-tap, tap-tap-tap, tap-tap-tap-tap, tap-tap-tap-tap-tap, and then back down tap-tap-tap-tap, tap-tap-tap, tap-tap, tap." He followed this by writing 123454321.

We each have our strategies in terms of what we see, hear, or feel, of getting out of bed in the morning, multiplying two numbers, deciding when to buy gas, or knowing that something is right. For example, Melanie in my pilot project repeatedly demonstrated a distinct problem solving strategy that lets her know that the result "is right." Let us look, for example, how she solved the following multiple choice problem: *Alana entered the county spelling bee. She spelled 47 words correctly before she made a mistake. If she had spelled three more words correctly, she would have spelled twice as many words as last year. How many words did she spell correctly last year?* A. *25* B. *27* C. *32* D. *35*

Here is how Melanie explained her solution (in terms of what she saw, heard or felt) in her homework: "I added each number to itself and 25 + 25 = 50. The problem says 47 then + 3 = 50. I did not feel anything but in my head I saw 47 + 3 = 50. I also saw that 50 was really gold and yellow and it was blinking and heard it beep. Beep, beep, beep, beep it sounded really fast and loud. My head was here [smiley face] and the numbers were here [smiley face below the first smiley face, shifted to the right, suggesting that she saw them in front, somewhat to the side]. The numbers were that big. The other numbers were black besides 50. The numbers were very clear. I saw the numbers for about a minute. I saw the numbers after the question. I saw the numbers in numbers not letters. The same thing happened with 25 + 25 = 50."

In my pilot project, I often asked the kids to "try on" each other's sensory strategies. By doing so, they were by comparison able to gain more awareness of their own strategies. I was amazed at the ease with which the kids adopted Melanie's decision strategy of seeing the correct answers blink.

Tools for separating the teacher/investigator's meaning from that of the student

Just as cognitive organisms can never match their conceptual and sensory organizations of experience with the structure of an independent objective reality because they simply do not have access to any such reality, so can we, teachers, never match the model we have constructed of the students' conceptualizations and sensory strategies with what actually goes on in their head. The best we can do is apply von Glasersfeld's principle of *fit* by constantly calibrating information and feeding it back to the students to test for accuracy and recognition, and accordingly adjusting our model*s*. How can we do this? How can we make sure that we separate our own meanings from those of the students?

For one, while attending to the students, we, as teachers, can pay attention to the communication *process,* not just the *content.* While content generally refers to *what* is talked about, or *why* it is talked about, process refers to the *how* of the way problems and solutions are communicated. Process, or pattern-based distinctions occur at different logical levels of communication than content-based distinctions do (Bateson 1972). Attending only to content makes it far more likely that the teacher will associate elements of the student's communications with her own private meanings rather than with the student's. By also attending to process rather than only to content, the teacher can detect order or pattern, using other ways of knowing besides rational logic, such as attending to physiological and language cues.

Although sensory experience is simultaneously available to all senses, people attend to various aspects of see-hear-feel experience at different times, which is manifested in their language. For example, let us take the case of two children trying to work together on a mathematics problem. One child does "not *see*" what they are supposed to do, while the other states she doesn't get "a *feel*" for what they are supposed to do. In this scenario, communication flow is obstructed because each child is attending to a different sense system, or logical level of experience (Bateson 1972). By noticing this, the teacher can help the children translate their experience so it can be shared and attention can again flow freely. Sensory system mismatches often take place between teachers and children. For example, if a child says, "Your explanation is somewhat *foggy*," the teacher's response of matching the visual system by asking "What would it take to make it *clearer*?" might be a better *fit* than the kinesthetic mismatch of "So you *feel* confused?"

People's sensory strategies are processes that cause "changes in body state – those in skin color, body posture, and facial expression, for instance – [which] are actually perceptible to and external observer." (Damasio 1994, p. 139). These physical reactions are important cues for the external observation and confirmation of people's sensory strategies. The primary behavioral elements involved are: language patterns, body posture, accessing cues, gestures, and eye movements (Dilts et al. 1980; Pasztor 1998; Hale-Haniff & Pasztor 1999).

Attending to the sense system presupposed in people's language is based on the assumption, derived from constructivist therapy case studies and literature, that sensory experience or "the report of the senses" reflects the interaction between body and mind, and that one can attend to communication behavior as a simultaneous manifestation of sensory experience. For example, constructivist therapies are particularly successful in using linguistic metaphors such as "That's a murky argument," "Things were blown out of proportion" or "Shrink the problem down to size" (visual); "This is an unheard of solution," "It has a nice ring to it" or "He talks in circles" (auditory); and "It feels right," "The solution hit me" or "This is hot stuff" (kinesthetic), as an expression of people's sensory experiences (Bandler & MacDonald 1988; Pasztor 2004b).

Most often, we do not need training to understand the language of behavioral cues. For example, if a person is using gross body movements – large motor movements compared to fine motor movements – we instinctively know what the relationship between level of detail and abstraction in the submodalities of his internal processing is. It would be really odd for that person to say, "I got the details, now give me the big picture." The more precise the body language, the more precise the "chunk size" of information the person is processing. We can also tell the high degree of detail by the narrowing of the gaze – it's almost as if the person was focusing on a particular area of the fine print as opposed on a diffused thing, like noticing a page or a computer screen. Duration and intensity of gaze, coordination of eye and head movements, head tilt and angle, chin orientation (up, down and middle) – some of these are *accessing cues*. They might tell us the state that people are in, the configuration of their attention, level of detail, what they are attending to. Sometimes people lean their head to one side when they are receiving new information, and to another side when it is "a rerun." Noticing these cues can be very helpful to see that the person is receptive to what we're saying or when their system is closing down a bit. In the latter case, how can we shift the way we are presenting information so that they open back up again?

Let us say a person wanted to learn the subject area and we noticed their physiology starting to shut out new information. Being able to map the precise point where they shut down and to figure out what was going on that caused them shut down can be helpful to facilitate their getting back in state.

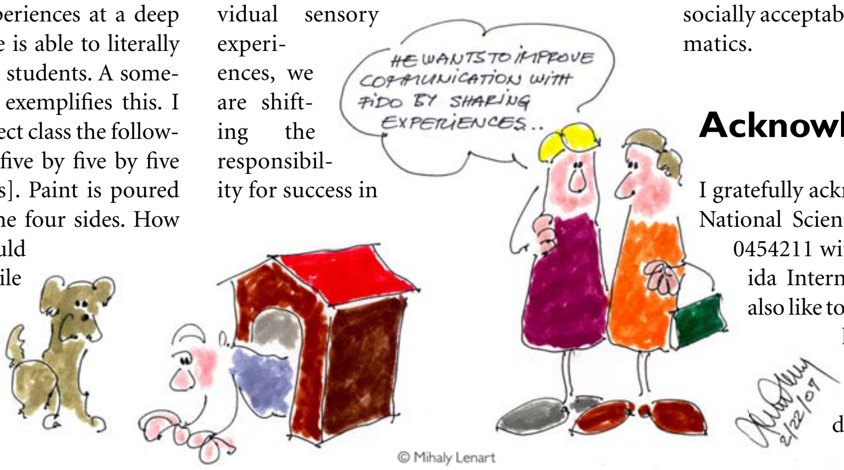
Awareness of behavioral cues also has the benefit of dispelling misconceptions that parents and teachers often have about the children's behavior. You have probably heard parents or teachers say to their children, "The answer is not on the ceiling!" while forcing them to look down on their notebooks when doing their homework or taking a test. In doing so they inadvertently keep the children from accessing information visually and instead lock them into the kinesthetic modality. This is of particular significance in mathematics, where visualization is often the key to solving a problem. You have probably also heard parents or teachers say to their children "look at me when I talk to you." When people listen, they have a natural tendency to turn their ear toward the sound source, so facing it will not come naturally to them. Sometimes we force our children to look at us while we talk, and then we complain that "you haven't heard a word of what I said, have you?" You have also probably heard parents or teachers say to their children, "Stand still when I talk to you!" While I do not have much room here to discuss behavioral cues in much detail, I want to emphasize that being able to recognize their correlation to internal processing might be a critical tool for helping someone access optimal learning states. It may also be all it takes to categorize a child as "gifted," as opposed to "at risk."

Democratization of math education: Utilizing SEL

The premise for utilizing SEL is that if the teacher *embodies* the distinctions of subjective experience that encompass SEL in her neurology and mindfully reflects them in her communication with the students, then she is able to share the students' experiences at a deep sensory level and thus she is able to literally "make more sense" of her students. A somewhat humorous incident exemplifies this. I presented to my pilot project class the following problem: "Imagine a five by five by five cube [made of unit cubes]. Paint is poured down over the top and the four sides. How many [unit] cubes would have paint on them?" While some kids said, "All," some others felt real confused. Upon eliciting their see-hear-feel experiences using the distinctions of SEL, I was able to understand that the kids who had said, "All," had imagined a thinner paint that got underneath the cube and into the cracks between the unit cubes, while the ones who felt confused, imagined the paint "too" thick and concluded that it my not cover the cube evenly enough to have whole unit cubes covered. Ultimately, I was able to separate students' images of the paint from mine, and thus realize that I had actually specified the problem poorly.

The key to utilizing SEL is to model students' subjective experience to help them amplify successful learning states by bringing them into consciousness, and, if necessary, to help them shift un-resourceful learning states so that they become resourceful.. The premise is that experiences are like a series of dominoes: the more dominoes are falling, the more difficult it is to break un-useful learning patterns. If we can find the first domino or what has knocked down the first domino, so to speak, then the person has much more choice than when his negative response – be it anger, frustration, or helplessness – is real high. It is much more likely that a student has choice while his response to a negative state of learning is still small, and it gives him a sense of control to be able to change it. Through the process of modeling students' experiences we can slow down their processing so they are able to gain conscious control over their sensory strategies and thus gain conscious mathematical competence.

By rooting mathematics understanding in each student's individual sensory experiences, we are shifting the responsibility for success in mathematics from the students back to those who guide and lead the process of co-constructing knowledge. This, in turn, should radically change prevailing beliefs about who should be studying mathematics and who should be successful at it: everybody has access to understanding, not just those who possess the "math gene" – it should not be socially acceptable anymore to fail in mathematics.



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