Radical Constructivism: A Scientific Research Program

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Purpose: In the paper, I discuss how Ernst Glasersfeld worked as a scientist on the project, Interdisciplinary Research on Number (IRON), and explain how his scientific activity fueled his development of radical constructivism. I also present IRON as a progressive research program in radical constructivism and suggest the essential components of such programs. ***Findings:*** The basic problem of Glasersfeld's radical constructivism is to explore the operations by means of which we assemble our experiential reality. Conceptual analysis is Glasersfeld's way of doing science and he used it in IRON to analyze the units that young children create and count in the activity of counting. In his work in IRON, Glasersfeld first conducted a first-order conceptual analysis of his own operations that produce units and number, and then participated in a second-order analysis of the language and actions of children and inferred the mental operations that they use to produce units and number. Further, Glasersfeld used Piaget's concept of equilibration in the context of scheme theory in a second-order analysis of children's construction of number sequences and of more advanced ways and means of operating in the traffic of numbers.
Research Implications: The scientific method of first- and second-order conceptual analysis transcends our work in IRON and it is applicable in any radical constructivist research program whose problem is to explore the operations by means of which we construct our conceptions. Because of the difficulties involved with introspection, conducting second-order conceptual analyses is essential in exploring these operations and it involves analyzing the language and actions of the observed. But conceptual analysis is only a part of the research process because the researchers are by necessity already involved in creating occasions of observation. The "experimenter" and the "analyst" can be the same person or they can be different people. Either case involves intensive and sustained interdisciplinary thinking and ways of working if the research program is to be maintained over a substantial period of time as a progressive research program.
Key Words: Scientific research program, attentional model, conceptual analysis.

It is easy to read Glasersfeld's texts on radical constructivism and interpret them as indicating that he regards radical constructivism as a finished model of knowing. However, after working with Glasersfeld for a rather extended period of time while he was developing his model of knowing, I fully believe that he does not regard radical constructivism in that way. Rather, my interpretation is that he regards it as a continually evolving model whose evolution is fueled and sustained by the novel scientific work of its adherents.[1] I feel justified in my belief because, as I worked with Glasersfeld, I learned that his scientific work indeed fueled and sustained his philosophical and epistemological inquiry. So, I was delighted that my account of how Glasersfeld regards radical constructivism is compatible with Alexander Riegler's characterization of the constructivist community at large as "a coherent and largely consistent scientific effort to provide answers to demanding complex problems" (Riegler 2005, p. 1).

I came to know Glasersfeld as a scientist through our intensive collaboration in the project, Interdisciplinary Research on Number (IRON). Through that collaboration, I experienced radical constructivism as a way of thinking and learning when doing science. It was a part of who we were and how we thought, and I still regard it as a living, growing model of knowing and learning. Glasersfeld captured the intensity with which we approached our work in IRON[2] in a passage in *Thirty Years Radical Constructivism*:

> "We had heated arguments and for all of us it was a powerful lesson, hammering in the fundamental fact that what one observer sees is not what another may see and that a common view can be achieved only by a strenuous effort of mutual adaptation." (Glasersfeld 2005, p. 10)

This passage points to the countless hours we spent trying to reach some semblance of a consensus concerning video-recorded material of children's numerical operating and the way in which we operationalized the basic tenets of radical constructivism. So it is natural for me to portray Glasersfeld as a scientist as well as to portray how his scientific work was a constitutive part of his development of radical constructivism.

In the following text, I provide a brief account of our interdisciplinary work on how children construct number. After that, I portray IRON as a progressive research program in the sense of Lakatos (1970) and explain how Glasersfeld's work was essential in constituting the program as progressive. I also suggest how our work in IRON contributes to other radical constructivist research programs whose central problem is to explore the operations that are involved in constructive activity.

Explaining how children construct number

The IRON project began in 1975 after Glasersfeld joined the Psychology Department at the University of Georgia in 1969 and after his colleague Charles Smock had introduced him to the work of Piaget. One might wonder why an epistemologist, mathematics educators, and a philosopher of mathematics would spend countless hours in heated arguments concerning children's construction of number. Perhaps the most salient reason is that explaining how children construct number is an extraordinarily compelling problem that has far-reaching implications for solving other such problems. In addition, it was our intention to establish a constructivist research program in mathematics education that included mathematics teaching as a central core. Also on our agenda was initiating a constructivist revolution in mathematics education to countermand the stranglehold that behaviorism had on the field in the United States after the demise of the modern mathematics movement.[3] But our most immediate goal was to synthesize our different ways of thinking for the purpose of constructing an explanatory model of children's construction of number.

We each brought results from our preceding work to our discussions in IRON. I brought an experiential model of children's numerical operating that was developed as a result of longitudinal teaching experiments and Glasersfeld brought his attentional model for the construction of units and number. Before we began the interdisciplinary work of IRON, Glasersfeld, being the passionate and consummate scholar that he is, had read a substantial portion, if not all, of the books by Piaget and his collaborators.[4] This was a fundamental factor in our work because it not only grounded our work in Piaget's genetic epistemology, it also grounded our work in the scientific work of the Genevans.[5]

It is crucial to point out that we each had our own purposes for engaging in interdisciplinary work. My educational purpose was to develop an itinerary for children's construction of number that would be useful in the mathematics education of children and Glasersfeld's scientific purpose was to "study how intelligence operates, of the ways and means it employs to construct a relatively regular world out of the flow of its experience" (Glasersfeld 1984, p. 32).[6] Although we collaborated on developing a model of how children construct number, the fact that we did have different purposes illustrates the power of our interdisciplinary collaboration.

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The preliminary work

An experiential model

Initially, I experimented with using Piaget's analysis of children's construction of number as a guide when working with children. Piaget's analysis had led him to the following position:

> "The development of number does not occur earlier than that of classes (classificatory structures) or of asymmetrical transitive relations (serial structures), but there is, on the contrary, a simultaneous construction of classes, relations and numbers." (Piaget 1966, p. 259)

His minimal criterion for children's construction of number was operative one-to-one correspondence that, in his model, was made possible by the emergence of the arithmetical unit. The stages in the construction of one-to-one correspondence exactly paralleled those of the construction of operative classification and seriation (Piaget 1966, 260ff), so I focused on classifying, ordering, and one-to-one correspondence as activities that might engender children's construction of number. Unfortunately, this effort proved to be of questionable value. So, I abandoned Piaget's analysis as a guide in how I operated and turned to teaching young children for rather extended stretches of time *so that the children* might teach me what their ways and means of operating with number might be like (Steffe, Hirstein, & Spikes 1976). It is of great interest to me now that others studying children's construction of reading and writing have taken a similar approach to how to make progress in children's education (Ferreiro 1991).

A major thing that I learned was that the activity of counting is children's primary means of solving their arithmetical situations. I also learned that there are major differences in the units that children create when counting. For example, when finding how many checkers were hidden by two cloths, one hiding seven and the other five, some children would sequentially put up seven fingers in synchrony with uttering "1, 2, 3, 4, 5, 6, 7," and then continue sequentially putting up the remaining three fingers in synchrony with uttering "8, 9, 10." They would then fold all fingers down and continue by sequentially putting up two more fingers in synchrony with uttering "11, 12" and then say, "twelve." This was in contrast with children who couldn't count unless they could actually see and/or touch the checkers, and in contrast with children who would simply start with "seven" and count on five more times (i.e., 7; 8-9-10-11-12"). There were other more advanced ways of counting as well as other variations of units.

The first way that I described above regarding how some children counted is indeed spectacular because the children created their own units to count [sequentially putting up fingers] when intending to count checkers. In fact, at the time I believed that the act of sequentially putting up fingers while counting was the first indication of the presence of Piaget's arithmetical unit. When children had to actually see and/or touch the checkers in order to count, these acts still indicated that the children were creating units to count while counting, but they were more immediately tied to their perceptual experience in counting. When children simply said, "seven," and then continued counting five more times, this was a more solid indication of Piaget's arithmetical unit. Piaget (1970) was well aware of the importance of units, but his characterization was restricted to arithmetical units and he did not provide an account of units at the sensory-motor level. For Piaget, arithmetical units are created when; "Elements are stripped of their qualities" (Piaget 1970, p. 37). He gave no account of children's construction of units that precede or follow arithmetical units nor did he give an account of mechanisms that do the stripping.

When I began working with Glasersfeld in 1975, as I suggested above, I had an extensive corpus of video-recorded material of children solving numerical problems as well as an experiential model of the units children create in counting activity. But I was yet to construct a theoretical model for how the mind makes the units that I observed the children making as well as a theoretical model for children's counting that explained the differences that I had observed. More importantly, I was yet to explain how children constructed number.

A hypothetical model of children's construction of units and number

Although Glasersfeld was never involved in the actual teaching experiments with children that I directed, he was highly engaged in the conceptual analyses[7] of an extensive corpus of the video-recorded material of children solving numerical problems. I will never forget late one morning circa 1978 when Glasersfeld declared to me that, "I now understand what mathematicians mean by a set!!" This event signaled a major breakthrough in our attempts to understand how the mind makes the units in counting that I had observed, and it occurred prior to the publication of his seminal paper on the conceptual construction of units and number (Glasersfeld 1981). In formulating his model, Glasersfeld drew on his work with Silvio Ceccato whom he credits as the first to interpret the structure of certain abstract concepts as patterns of attention (Ceccato 1974). According to Glasersfeld,

> "Attention is not to be understood as a state that can be extended over longish periods. Instead, I intend a pulselike succession of moments of attention, each one of which may or may not be 'focused' on some neural event in the organism. By 'focused' I intend no more than that an attentional pulse is made to coincide with some other signal (from the multitude that more or less continuously pervades the organism's nervous system) and thus allows it to be registered. An 'unfocused' pulse is one that registers no content." (Glasersfeld 1981, p. 85)

Glasersfeld's model of pulsating moments of attention provided an explanation of the mental operation that is involved in the construction of ordinary items of experience and the role these items play in the construction of numerical units. He called the operation "unitizing." A group of co-occurring sensory-motor signals becomes a "whole" or "object" when an unbroken sequence of attentional pulses is focused on these signals and the sequence is framed or bounded by an unfocused pulse at both ends. The unfocused pulses provide closure and set the sequence of contiguous focused pulses apart from prior and subsequent attentional pulses.

A focused moment of attention registers sensory material and an unfocused moment of attention can be regarded as a blank space. The records of making a sensory-motor item, or an item of experience, were graphically illustrated in terms of an *attentional pattern* as shown in Figure 1 (Glasersfeld 1981, p. 87).

Figure 1: An attentional pattern: Sensory-motor item

The unfocused moments of attention are designated by "O" and bound the focused moments of attention designated by "I." The letters a, b, … , k designate sensory material selected by attention and this sensory material is registered as records of experience. I emphasize that the *attentional pattern* or *recognition template* is established as a result of individual–environment interaction and the process it symbolizes constitutes a model of the operation that is involved in compounding sensory-motor signals together in the immediate here-and-now to form items of experience – the unitizing operation.

Sensory-motor items are isolated in experience and there may be no element of recognition in their establishment. If a child does recognize an experiential item as having been experienced before, this constitutes the beginnings of categorizing items together. Categorizing, however, goes beyond the simple recognition of a sensory-motor item. It involves a sense of similarity in the way explained by Inhelder & Piaget (1964) when distinguishing graphic and nongraphic collections.

> "We use the term "collection" rather than "class" in the strict sense, because the former term carries no implication of a hierarchical structure of class-inclusions. However, these collections are no longer graphic, and objects are assigned to one collection or another on the basis of similarity alone. (p. 47)

Assigning objects to a collection in this sense requires an abstraction beyond the abstraction that is involved in recognizing a particular sensory-motor item. It involves more because sensory-motor items are formed in the moment and there may be no recollection of a preceding experience even though a current item may be recognized. Categorizing sensory-motor items together involves recollection of previously experienced items,[9] and re-focusing attention on the items is an act of taking them together, a process that Glasersfeld called reprocessing or attentional iteration.

Reprocessing sensory-motor items encourages focusing attention on the unitary wholeness of each item, which is an operation of unitizing the sensory-motor items. The unitary item produced is diagrammed in Figure 2 (1981, p. 89).

Figure 2: The attentional structure of a unitary item

Glasersfeld used the notation in Figure 2 to designate a single attentional moment focused on the unitariness of a sensory-motor item. In this, n is used to denote the necessity of having some, but no particular, sensory-motor material on which to focus. This development of the unitizing operation opens the possibility of the child categorizing non-homogeneous items together on the basis of their unitariness or wholeness – "things" that go together because they *are put together.* Glasersfeld referred to these types of collections as lots to indicate that the categorization was not constrained to any particular perceptual material.

Reprocessing the items of lots encourage stripping sensory content from the unitary items, which produces an abstract unit item notated in Figure 3 on the next page (Glasersfeld 1981, p. 91).

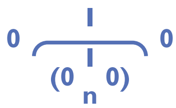

Figure 3: The attentional structure of the abstract unit item.

The construction of the abstract unit item opens the possibility of using that item to recursively reprocess the items of lots. This recursive reprocessing can produce an attentional structure called a composite unit – a unit of abstract units – diagrammed in Figure 4.

0(0I0 0I0 0I0 0I0 0I0)0

Figure 4: The attentional structure of a composite unit.

I have chosen to portray the composite unit in Figure 4 as containing only five abstract unit items to emphasize the role that figurative patterns play in the initial construction of composite units containing five or fewer items (Glasersfeld 1982b). These structures are what Glasersfeld meant by the emergence of sets in children.[9]

Even though the above development of the attentional model is not complete, it does portray what Glasersfeld brought to the scientific work on children's construction of number. After explaining a series of abstractions that produced these items, he commented that:

> "It now remains to be seen whether this model provides a new and more successful approach to an understanding of the still problematic activities of counting and the operations involved in establishing specific numerosities." (Glasersfeld 1981, p. 94)

The preceding quotation captures the initial thrust of our collaborative scientific work in IRON, where our attempt was to use what was then an evolving version of Glasersfeld's hypothetical model in constructing the unit items that children create as they count. An indispensable step in constructing a model of children's counting types involved our recursively returning to critical segments of the videotaped material that I had developed. This was an indispensable step because Glasersfeld's attentional model constituted a new way of thinking about how children construct units of all kinds, so a continual reinterpretation of the critical video segments was essential. The conceptual analysis in which Glasersfeld engaged to produce the attentional model was extraordinarily insightful scientific work, and his collaborative conceptual analysis in constructing a model of children's counting types was every bit as insightful.

A model of children's counting types

Glasersfeld wasn't as interested in his interpretations of the children's mathematical activity as he was in how the others in the project thought about the children's mathematical activity. He did make his own interpretations of the children's numerical ways and means of operating, but he always checked his interpretations with the other members of the interdisciplinary project, and this was the source of our heated arguments.[10] Through our collaborative work, I learned that conceptual analysis is an essential part of doing science in mathematics education and I have used this kind of analysis in all of my subsequent work in IRON and beyond.

Through our discussions, an important insight came to the fore that opened the way for major progress. The insight was that children create the items they count in the activity of counting. This is clearly demonstrated when a child appears to count specific items in a situation where no items of that kind are within the child's perceptual field in the way that I described children establishing putting up fingers as unit items when counting hidden checkers in my experiential model. Creating motor unit items as substitutes for hidden unitary items is the culmination of a rather intricate development in the child. Involved in that development is the progressive ability to create unit items on the basis of, first, visual, auditory, and tactual perception, and then proprioceptive sensation (Steffe, Glasersfeld, Richards, & Cobb 1983, p. 116).

Creating perceptual unit items

When children use their unitizing operation recursively to unitize sensory-motor items, as I noted above, Glasersfeld used "unitary items" to refer to the result. In the model of counting types, we decided to call the unit items that children create when counting unitary items *perceptual unit items* to distinguish them from the unitary items. We also called children who are restricted to creating perceptual unit items when counting unitary items *counters of perceptual unit items.*[11]

That some children of even six years of age are restricted to creating perceptual unit items in counting was not anticipated by Glasersfeld's model of units and number, nor was the abstraction that is necessary for children to make the next step in creating items in counting when the unitary items are hidden from the children's view. But it did provide an essential tool to explain how children use motor acts like putting up fingers, hitting their desk with a pencil, or some other relevant motor act as countable items that are substitutes for hidden unitary items that they intend to count.

Creating motor and verbal unit items

When children use their unitizing operation to unitize the proprioceptive sensation that is involved in counting, we called the results of the operation motor unit items. It was necessary to use children's capability to produce visualized images of unitary items to explain how children substitute motor unit items for hidden unitary items that they intend to count. It seemed natural to call the re-presented unitary item a figural unit item and characterized the substitution as occurring at the level of re-presentation. We also called children who are restricted to creating motor unit items as substitutes for hidden unitary unit items when counting *counters of motor unit items.*

Counters of motor unit items always start counting with "one" as do counters of verbal unit items, which is the next unit item we isolated in children's counting. When children unitize the vocal productions when counting motor unit items, the vocal productions come to be used as substitutes for the motor unit items. The motor unit items are dropped out because simply saying a number word signifies them.

Creating abstract unit items

The abstract unit item was the next countable unit item in the progression of the types of units children create while counting. Counting-on, as I describe below, is the behavioral indicator of the ability to use the abstract unit item in acts of counting. Given, for example, the following task: "There are seven marbles in this cup" (rattling the marbles in the cup). "Here are four more marbles" (placing four marbles on the table). "How many marbles are there in all?" Glasersfeld analyzed counting-on as follows.

> "If the child says there are seven in the cup, and proceeds to count the additional marbles, "8, 9, 10, 11 – eleven!" it suggests that in uttering "seven" the child knows that the number word, in the given context, stands for a specific collection of individual unitary items that satisfy the template called 'marble' and that, if counted, they could be coordinated with utterances of the number words from 'one' to 'seven.' The child knows this and therefore does not have to run through the activities that would actually implement it on the level of sensory-motor experience." (Steffe et al. 1983, p. 42)

Such counters can also mentally "run through" counting activity starting with other number words and count so many more times. They can turn anything whatever into countable items because counting has become a reflective process in Piagetian terms and, as such, it is "operative" rather than "figurative".[12]

Fueling and sustaining the radical aspect of radical constructivism

Even though the above discussion of our work on counting types is necessarily brief, I have sketched some of the results of Glasersfeld's scientific work on matters of crucial importance in the mathematics education of children as well as in radical constructivism. It is important to point out that our work formed an essential connection with Piaget's genetic epistemology (Glasersfeld 1982a). In a paper that represents his early analysis of Piaget's genetic epistemology (Glasersfeld 1974), Glasersfeld provided an extensive discussion of Piaget's research that undermines the belief that, the knower and the things of which, or about which, he or she comes to know are, from the outset, separate and independent entities. The basic research that he drew from is Piaget's account of the child's construction of the concept of an object that has some kind of permanence in his stream of experience (Piaget 1955). In the paper, Glasersfeld portrays how the infant comes to be but one element or entity among others in a universe that he or she has gradually constructed for him-or herself out of the elementary particles of experience. This powerful insight into the child's construction of his or her ordinary items of experience serves as a justification that, from the outset, the knower and the things of or about which he or she comes to know are separate and independent entities is not viable.

The connection between Glasersfeld's model of the unitizing operation and Piaget's account of the child's construction of experiential reality resides in the realization that the unitizing operation provides an opening to study mental operations of the mind that are involved.

> "Radical constructivism maintains – not unlike Kant in his Critique – that the operations by means of which we assemble our experiential world can be explored, and that an awareness of this operating … can help us do it differently and, perhaps, better." (Glasersfeld 1984, p. 18)

Our work using the attentional model in specifying the types of units children use in counting constituted an exploration of those operations of the mind that are involved in assembling not only experiential reality, but mathematical reality as well. Husserl also proposed, "that the mental operation that unites different sense impressions into the concept of "thing" is similar to the operation that unites abstract units into the concept of number" (Glasersfeld, 2006, p. 65). So, given that children construct what Glasersfeld calls *experiential* realities out of elementary particles of experience, one can infer that children construct their mathematical realities using their experiential realities. That is, one can infer that a child's mathematics is abstracted from his or her experiential reality and it is not given from the outset as an entity independent of the child. This analysis definitely served in fueling and sustaining the radical aspect of radical constructivism:

> "Radical constructivism is, thus, radical because it breaks with convention and develops a theory of knowledge in which knowledge does not reflect an "objective" ontological reality, but exclusively an ordering and organization of a world constituted by our experience." (Glasersfeld 1984, p. 24)

Radical constructivism as the core of scientific research programs

Glasersfeld's work in IRON demonstrates how radical constructivism can constitute the core of a scientific research program.

> "All scientific research programmes may be characterized by their '*hard core*.' The negative heuristic[13] of the programme forbids us to direct the modus tollens at this 'hard core'." (Lakatos 1970, p. 133)

Lakatos used "hard core" but I prefer to use just "core" because the adjective "hard" can indicate a non-changing core although this is not what Lakatos intended: "The actual hard core of a programme does not actually emerge fully armed … . It develops slowly, by a long, preliminary process of trial and error" (p. 133). Although I wouldn't say that it has changed by trial and error, it is indeed the case that the core of IRON has changed since it began.

First- and second-order analyses

Glasersfeld's model of units and number definitely should be a part of the core of any radical constructivist research program whose goal is to explore the operations by means of which we construct our conceptions. When I consider the architecture of the units and composite units that we produced when constructing the counting-types model, and the operations that children perform using these units as input,[14] I consider the counting types model as modifying the attentional model. For example, the concept of a figural unit item arose in analyzing children's counting behavior and this unit is not presented in the attentional model. The motor and verbal unit items were also not presented. These unit types certainly are not restricted to children's counting [e.g., sign language]. So, there is a reciprocal relationship between the attentional model and the counting-types model in that essential elements of the latter become knowledge of the analyst that can be used in further analyses.

Glasersfeld produced his model of units and number by using mental operations to analyze his own conceptions of units and number. So, I refer to his analysis as a first-order analysis. The goal of a first-order analysis concerns specifying the mental operations that produce particular conceptions of the analyst. It is an analysis of first-order models, which are models the analyst has constructed to organize, comprehend, and control his or her experience; that is, the analyst's own knowledge. The distinction I am making between the mental operations that produce particular conceptions of the analyst and those conceptions is crucial in understanding how the knowledge of researchers can be used in radical constructivist research programs concerned with exploring the operations by means of which we construct our conceptions. It is crucial because these operations are involved in producing second-order models.

When the goal is to explore operations by means of which human beings construct mathematics, following Piaget (1970), this involves exploring children's constructive activity.[15] In the exploration, we construct second-order models, which are models an observer constructs of the observed person's knowledge in order to explain their observations (Steffe, et al. 1983, p. xvi). Because the goal of the analyst in constructing second-order models concerns constructing conceptual operations that explain the observed language and actions or interactions of the observed person, I refer to it as a second-order analysis. The analysis in which we engaged to produce the counting-types model was a second-order analysis in which we used the results of Glasersfeld's first-order analysis. The reciprocal relationship between first- and second-order analyses is basic in radical constructivist research programs because it illustrates that researchers and their ways and means of operating and observing constitute the research programs.

The fundamental principles of the core

The fundamental principles of radical constructivism that Glasersfeld presented in 1989 were already a defining a part of the core of IRON even though the principles had not been as explicitly stated.

> "1. Knowledge in not passively received but built up by the cognizing subject.
> "2. The function of cognition is adaptive, and serves the organization of the experiential world, not the discovery of ontological reality."

When I first read these principles, I distinctly recollect how they encapsulated much of the content of radical constructivism that was known to me at that time. The second principle, of course, is a restatement of the radical aspect of radical constructivism. I mention them because they cut across radical constructivist research programs of all kinds.[16] But I do not regard the fundamental principles as explanatory principles in the way that we used the attentional model for units and number in constructing explanations of the units that children create in counting. Rather, researchers construct models in what Lakatos called the protective belt of the core of the research program that corroborates the core principles, such as Glasersfeld's analysis of Piaget's research on children's construction of permanent objects and our analysis of children's counting types.

A progressive research program

Although the book *Children's counting types: Philosophy, theory, and application* was a landmark publication in IRON, we only used those core principles that were useful to us in the work that produced the book. Essentially, after the publication of this book, we still had not explained how children construct number sequences. We had explained the units that children create in the activity of counting and hypothesized that they formed a developmental progression, but we had not explained children's counting in terms of number sequences nor had we specified the accommodations that produced the number sequences. So, I retreated into mathematics education and Glasersfeld retreated into his theoretical work. I launched a new teaching experiment because, in mathematics education, building experiential models precedes building theoretical models.

At the same time as I began the teaching experiment, Glasersfeld was fortunately working on a paper in which he interpreted Piaget's concept of equilibration and the two activities that constitute it, assimilation and accommodation, in the context of scheme theory. There was no agreement between the two of us that we would work on mutually compatible problems, but I wouldn't say that it was fortuitous either because we were both concerned with exploring operations by which we construct our experiential worlds. In what I consider as one of Glasersfeld's most important papers for mathematics education, he commented that:

> "Piaget's conception of assimilation and accommodation remain incomprehensible unless it is placed within the framework of his theory of knowledge and, specifically, into the context that he calls *schéme*. "Schemes" are basic sequences of events that consist of three parts. An initial part that serves as trigger or occasion. … The second part, that follows upon it, is an action … or an operation … . These two are, as a rule, explicitly mentioned when schemes are discussed. The third part is often only implied, but that doesn't make it any less important: it is what I call the result or sequel of the activity."[17] (Glasersfeld 1980, p. 81).

Analogous to the collaborative scientific work that produced our explanation of children's counting types, Glasersfeld's conceptual analysis of scheme, assimilation, accommodation, and perturbation[18], served in our collaborative scientific work in accounting for children's construction of number sequences after the teaching experiment had concluded. Five stages emerged in the constructive activity – two pre-numerical counting[19] schemes and three distinctly different numerical counting schemes[20] along with an explanation of the transitional accommodations (Steffe, Cobb, & Glasersfeld 1988). We also accounted for children's construction of adding and subtracting schemes within each of the stages. These number sequences quickly became part of the explanatory constructs of the IRON research program and were subsequently used to explain children's construction of multiplying and dividing schemes (Steffe 1994).

Although Glasersfeld retired from the University of Georgia in 1987 and joined the Scientific Reasoning Research Institute at the University of Massachusetts, that did not stop our scientific collaboration. After the work with children's construction of multiplying and dividing schemes, another problem shift occurred in IRON that became known as the reorganization hypothesis: *children's fractional schemes can emerge as accommodations in their numerical counting schemes*. Glasersfeld worked as a consultant on this project throughout its duration and was more than tangentially involved in the conceptual analyses of the video-recorded material that was produced when teaching children fractions. That the hypothesis was confirmed (Olive 1999; Olive & Steffe, 2002; Steffe, 2002; Tzur 1999) is, according to Lakatos, essential to claim that the IRON research program is a progressive program because the confirmations involved novelties not predicted by the hypothesis.

> "Finally, let's call a problem shift progressive if it is both theoretically and empirically progressive, and degenerating if it is not." (Lakatos 1970, p. 118)

The explanatory constructs of IRON were continually expanded to include an organization of schemes of action and operation that I call the mathematics of students. Aspects of these schemes would be explanatory principles of other radical constructivist research programs that explicitly include dynamic equilibrium, assimilation, accommodation, and scheme theory because they elaborate Glasersfeld's concept of scheme in significant ways such as the counting-types model elaborated his attentional model of units and number. Toward that end, the architecture of a scheme presented by Glasersfeld does not include, for example, reversible schemes, recursive schemes that take the scheme as its own input, schemes that are the result of coordinating a more basic scheme with the operations involved in producing a unit of unit of units, or schemes that function at differing levels of interiorization. So, rather than present the numerical schemes of the IRON research program per se as a part of the explanatory constructs of compatible research programs, the schemes' architecture is what is relevant.

Final comments

I have tried to say enough to portray IRON as an evolving and changing scientific research program. From the time we started to work together onwards, Glasersfeld developed the fundamental principles of radical constructivism concurrently with using them in interdisciplinary scientific work. Although the results of his analytical work proved to be of more immediate use than the results of his philosophical and epistemological work, the second fundamental principle of constructivism is essential in establishing what I think of as a constructivist school mathematics.

The primary difference between a constructivist and a conventional school mathematics resides in one's conception of school mathematics. In the latter case, school mathematics is regarded as a thing-in-itself independently of human thought and experience and, in the former case, school mathematics is constituted by the results of conceptual analyses, which are models of mathematical thinking and learning. These models consist of an organization of mathematical schemes of operation and the accommodations of these schemes that children produce as a result of interactive mathematical communication. The models can open possibilities for mathematics teachers to construct their own school mathematics in conjunction with their children. In fact, the models should be regarded as providing possibilities for teachers to explain their students' mathematical language and actions and for teachers' goal setting. Thinking of a constructivist school mathematics as a dynamic organization of mathematical schemes of operation in the mental life of teachers casts mathematics education as a very exciting field and marks it as an evolving and changing professional practice. Glasersfeld has said in many places that radical constructivism doesn't tell you what to do. His comment marks an essential attitude in how radical constructivism is used. One does not simply *apply* radical constructivism. Rather, one builds *living models* of radical constructivism that do not countermand its basic principles such as a constructivist school of mathematics.

The members of the IRON research program did indeed set a revolution in motion in school mathematics and Glasersfeld was at the vortex of that revolution. But the influence of constructivism manifest in professional recommendations for reform in mathematics education primarily concerned the first principle of radical constructivism without consideration of the second principle (National Council of Teachers of Mathematics 1989, 2000). As a result, recommendations for what was to be taught were not based on the second principle. But the constructivist revolution in mathematics education has not run out of steam, and as long as radical constructivist research programs like the IRON program remain in a progressive phase, these research programs will serve to sustain and intensify the revolution. That is the legacy of Ernst von Glasersfeld's work in mathematics education.

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Notes

1. Here, I am speaking of radical constructivism on the intersubjective level (cf. Glasersfeld 1995, p. 120).
2. The philosopher, John Richards, was an initial member as well as Patrick Thompson and Paul Cobb who were then doctoral students.
3. See Steffe & Kieren (1994) for an historical account of the rise of constructivism in mathematics education and the zeitgeist in which we operated.
4. In fact, Glasersfeld said that he used "radical" for the first time in a paper in which he interpreted the epistemological aspects of the work of Piaget (Glasersfeld 1974).
5. I use "Genevans" to refer to not only to Piaget, but to his collaborators as well.
6. John Richard's purpose was to reconstitute the philosophical foundations of mathematics.
7. Conceptual analysis was imported by Glasersfeld through his work with Ceccato and widely used in the IRON project. A conceptual analysis is an analysis of what might constitute the mental operations of others. When used to explain children's behavior, the aim is to produce thick descriptions of conceptual operations that, were children to have them, might result in them thinking in the way they do (Thompson & Saldanha, 2003).

8. When an attentional pattern is used in categorizing sensory-motor items together, Piaget's (1955) studies on object permanence indicate that children have developed the capability to use attentional patterns to produce visualized images of sensory-motor items that they can recognize (Glasersfeld 1995). These images provide the child with an awareness of an experiential item apart from its location in immediate experience.
9. I have only underscored the emergence of composite units. Children must learn to use these nascent structures as input for operating further, such as disembedding a subpart from a composite unit, joining two composite units together, removing items from composite units, etc.
10. Our attempts to establish intersubjective agreement were very difficult because the counting types model came to supersede the experiential model at a higher level of abstraction with new organization and structure.
11. To find how many checkers are in a lot of checkers, these children need to actually see or feel the checkers in order to create perceptual unit items in counting.
12. Children do not count abstract units per se. Rather, they use the records of experience in their abstract unit item to produce a figurative or a sensory-motor unit items in an act of counting.
13. The paths of research to be avoided.
14. A composite unit, similar to an arithmetical unit, implies an ensemble of operations that produce a unit of units.
15. Piaget attempted to explain the construction of mathematics developmentally in his program of genetic epistemology. The work in IRON is similar with the exception that the work is embedded in mathematics teaching and learning as well as in development.
16. IRON wasn't the only research program influenced by Glasersfeld. He has also worked in family therapy (Steffe & Gale 1995), science education (Larochelle, Bednarz, & Garrison 1998), and psychotherapy (Kenny 1988), among others.
17. Schemes can be interpreted as negative feedback loops (Glasersfeld 1995).
18. Dynamic equilibration, assimilation, accommodation, perturbation, and scheme are all core principles.
19. We called these two counting schemes the perceptual counting scheme and the figurative counting scheme.
20. We called these numerical counting schemes the initial, tacitly nested, and explicitly nested number sequences.

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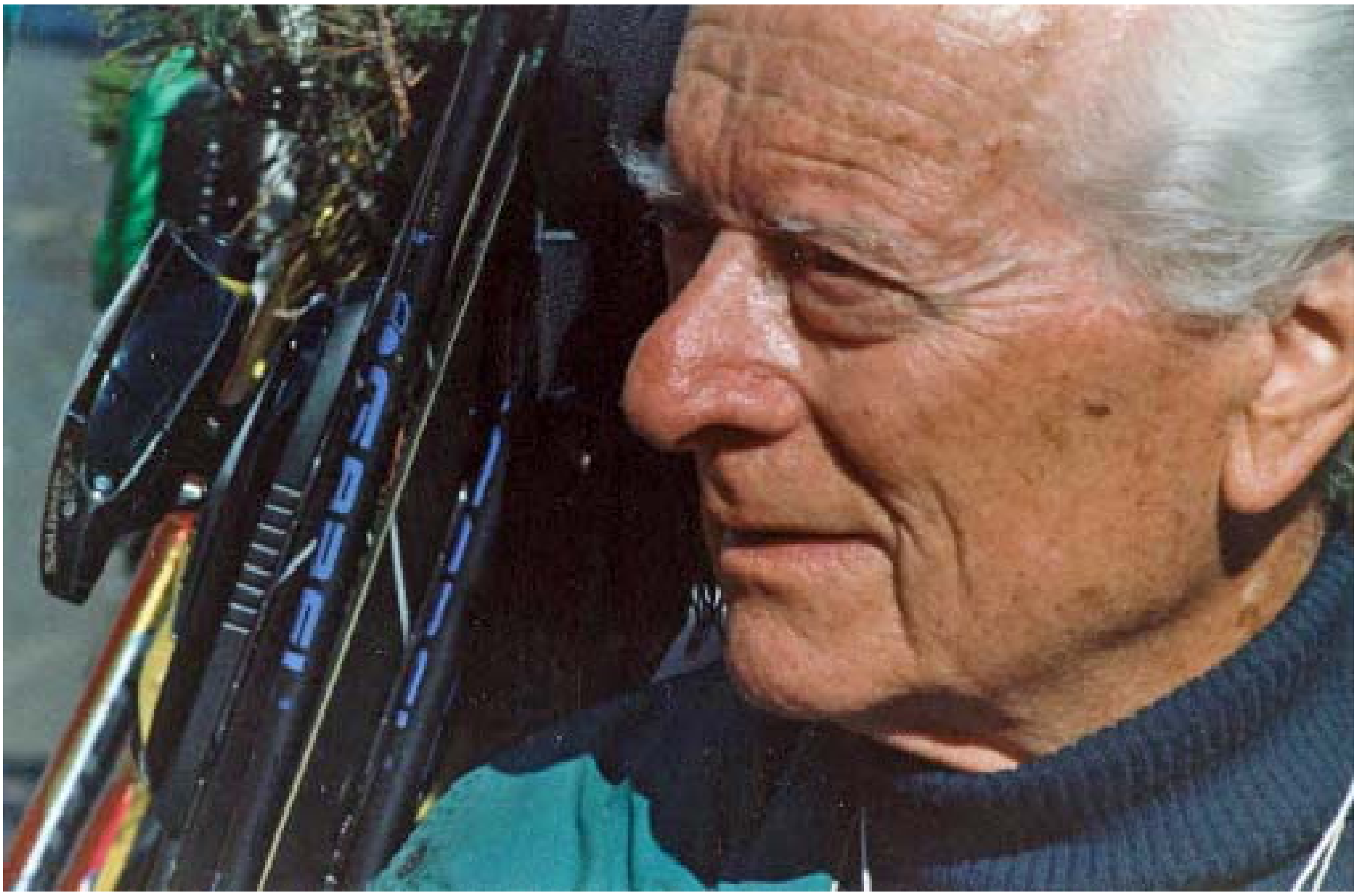

Ernst von Glasersfeld holding skis at St. Anton (Austria) 1999 (photo by Jack Lochhead).

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