An Analysis of the Recursive Structure of Problem Solving Activity

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> Abstract • I present an example of a student engaged in mathematical problem solving, to document the usefulness of the ideas discussed in the target article. Specifically, the example highlights (a) The recursive structure of the student's actions throughout her problem solving from initial meaning-making and formulation of the problem situation to her eventual solution; and (b) How the student's continuous structure of her problem solving (the continuum) makes possible a chain of particular cognitive expressions (the discrete), each of which provides an opportunity for the student to reorganize her prior actions.

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« 1 » The target article by Paolo Totaro and Domenico Ninno reminds us of the important role recursion plays in how living systems respond to perturbations from the environment. When triggered by perturbations from the environment, biological processes operate on themselves by replacing the altered components with other self-produced components that may be structurally different from the former yet remain organizationally equivalent (§3).

« 2 » Research in mathematics education has long considered recursion to play an important role in the construction of new knowledge. According to Jeremy Kilpatrick,

> “The learning of mathematics proceeds in a rhythm in which repetition is combined with variation. The repetition can take on the quality of recursion when old knowledge is used as a substrate for the construction of new knowledge.” (Kilpatrick 1986: 8f)

For example, the van Hiele model of geometry learning explains geometry learning as a recursive process, asserting that as students learn geometry, they move from a lower to a higher level of understanding, returning to the same geometric concepts with a different language that gives new meaning to the concepts and makes explicit what was previously implicit (Burger & Shaughnessy 1986).

« 3 » In addition to its role in geometry learning, recursion plays an important role in the actionbased accounts of mathematical performance that are based on the learning principles of Piaget's genetic epistemology (Glasersfeld 1991; Steffe 1994) and the applied studies of mathematics learning and problem solving (Cobb 1985; Freudenthal 1991; Cifarelli 1998; Cifarelli & Sevim 2015). For example, Paul Cobb considers the meaningmaking actions of problem solvers in terms of recursive activity, stating that –

> “the structure of a problem for the solver is determined by the schemes into which it is assimilated. The ensuing problem solving activity can then be viewed as an attempt to fill out or express this structure in a specific situation.” (Cobb 1985: 112)

« 4 » The authors of the target article address recursion of observation and how it operates as the norm that holds together the continuum and the discrete on one side and biological processes on the other (§2). Employing Maturana and Varela's (1980, 1987) characterization of recursion (§3), Totaro and Ninno provide illustrative examples from set and number theories that formalize the working connections and co-dependence between the continuum and the discrete (§§13–16) and thus make a persuasive argument of how the concept of recursion provides a key interpretative framework for phenomena of life and cognition.

« 5 » In my commentary, I discuss how in mathematics performance we can view the student's evolving reflections in problemsolving situations as a recursive process that initiates and provides an ongoing interpretive framework for the student's actions. In the following paragraphs, I describe the problem solving of a graduate student, with the pseudonym Sarah, from a university in the southeast United States. I present episodes of her work solving a Number Array task (Figure 1). This task provides students with opportunities to explore and develop mathematical relationships within an openended problem situation. My purpose is to

Find as many relationships as possible among the numbers

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 1 • Number Array task (from Becker & Shimada 1997).

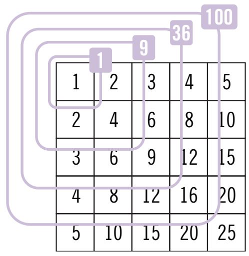

Figure 2 • Sarah's markup of the array.

illustrate how, through recursion and reflection on her actions, Sarah's problem solving evolved from initial meaningmaking and problem formulation to more mathematically sophisticated processes. In terms of the target article, the result of her initial reflections was that she was "seeing" (§14) and focusing her attention on a particular problem relationship within the array. Her initial sensemaking served as a foundation upon which to build a more mathematically sophisticated understanding of the problem she had formulated. Further advances resulted from her repeated selfreproductions of her solution activity, which she could reflect on or "look" (§14) at results of her prior explorations.

« 6 » Many mathematical relationships can be constructed based on the symmetry of the array. For example, in any rectangular block, the products of the opposite corner numbers are equal. So, for the block $\begin{smallmatrix}1 & 2 & 3\\ 2 & 4 & 6\end{smallmatrix}$ we have $1 \times 6 = 2 \times 3$. If the block is a square, the products of numbers on each of the two diagonals are equal. So, for the square block $\begin{smallmatrix}2 & 3 & 4\\ 4 & 6 & 8\\ 6 & 9 & 12\end{smallmatrix}$ we have $2 \times 6 \times 12 = 6 \times 6 \times 4$.

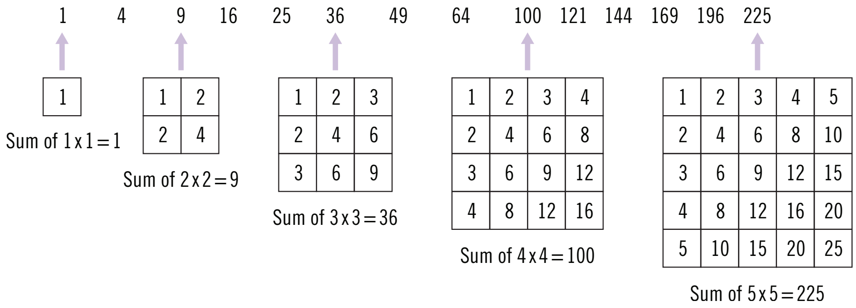

Figure 3 • Sarah's skipping method.

1	2	3	4	5	6	21
2	4	6	8	10	12	42
3	6	9	12	15	18	63
4	8	12	16	20	24	84
5	10	15	20	25	30	105
6	12	18	24	30	36	126

Sum of rows: 21 (1 + 2 + 3 + 4 + 5 + 6) = 21(21) = **441**

Skipping in the sequence:
Finding the sum for the 6 x 6 block

225 256 289 324 381 400 **441**
15^2 16^2 17^2 18^2 19^2 20^2 **21^2**

Figure 4 • Sarah's computation of the sum of entries in a 6×6 block.

« 7 » For the purposes of this commentary, I will focus on how Sarah problematized the situation by exploring a relationship between the sequence of square numbers on the diagonal of the array (1, 4, 9, …, 81, 100) and sums of entries in N × N blocks running down from left-to-right in the array. Sarah started by marking up the array with her pencil to generate a sequence of N × N blocks that included the square numbers on the diagonal (Figure 2).

« 8 » Sarah's initial actions constitute an immediate internal experience that is consistent with first-level "seeing" (§16) and at the same time establishes an organization that she recursively reproduces and builds upon throughout her problem solving. Specifically, she saw that she could find the sums by skipping over numbers in the diagonal sequence of square numbers (Figure 3):

"For a 1 × 1, I get a sum of 1. For the 2 × 2, I get a sum of 9 … but what happened to 4? It has been skipped! Okay, let me try this, I will write down the sequence of squares of all numbers, all in a row. So, the first number, 1, tells the sum of the very first matrix, a 1 × 1. And the first 2 × 2 has a sum of 9. So, I skipped over 4 to get the next sum, going from 1 × 1 to a 2 × 2, a sum of 9. The 4 gets skipped. Interesting! For the first 3 × 3 block, the sum is 36. In going from the 1 × 1 to the 2 × 2 to the 3 × 3, we go from 1 to 9 to 36. So, we skipped over the 16 and the 25, a skip of 2 in this sequence!! If this is true, then it looks like we will skip over the next 3 square numbers, and then the sum for a 4 × 4 should be 100. Cool! So, for a 5 × 5, we skip over the next 4 numbers in the sequence and get 225."

« 9 » Sarah looked to make sense of her skipping method with some further exploration. She extended her original sequence beyond 225, crossed out the corresponding skips and got a result of 441 for the sum of entries in the 6 × 6 block (Figure 4) and then looked to find out why her skipping method worked. As a result, she changed the structure of her problem solving from iteration of the skipping method to determining reasons the skipping method worked:

"I wonder why this skipping works. Let's see it another way, for the 6 × 6, we add the entries in the rows to get 21 + 42 + … + 126 = 21(1 + 2 + 3 + 4 + 5 + 6) = 21 × 21 = 441. Do we get 441 by skipping the next 5 in the square sequence? But I also notice that 21 over here is the sum of the first 6 numbers in that first row. Yes!"

"So, to find the sum of N × N blocks, I bet you just need to look at the sum of 1 to N and then square that total to get the sum. Let's try an 8 × 8. So, 1 + 2 + … + 8 = 36 and then I take 36^2? That comes out to be 1296. Does it check with my skipping over here? Let's see, I first skip 6 over 21 to get 28^2 for 7 × 7, and then skip 7 more to get the one for 8 × 8, so 7 more is 35, and the next one is 36! My algorithm works! The algorithm is efficient for large numbers beyond all these – how about a 100 × 100 grid! – But I thought that the skipping relationship was cool!"

« 10 » In sum, Sarah's problem-solving actions included not only informal interpretations, algorithms and mathematical relations, but also idiosyncratic diagrams (Figure 2) and small explorations of why things work, the results of which were not always correct or incorrect. They served as generated data that Sarah could reflect on and build upon as she looked to learn about the properties of the problem she formulated from the array. In addition, once Sarah derived from her explorations an initial understanding of the task, she had a conceptual grasp that could then be re-formulated as she considered more sophisticated properties of the relationships. Researchers have noted how the solver's consideration of these seemingly minor problems often impacts their changing views of the problem situation, which may lead them to re-formulate their goals and purposes and thus impact the problems they solve (Steffe 1988; Lave 2016).

« 11 » Sarah's activity is an illustration of the recursion of observation described by Totaro and Ninno (§39) in that each of her problem-solving actions transformed her prior actions into results to further reflect on. By always reflecting on the results of her prior actions, Sarah's problem solving became increasingly more sophisticated at each step of the process, both in terms of her generating more mathematically advanced algorithms and getting closer to her goal of discovering more mathematical relationships within the given array. In transitioning from an initial idea of skipping to an algorithm, she had a more mathematically powerful way to solve extensions of her problems beyond the 10 × 10 array to remark on how she could now solve a 100 × 100 array. In this way, Sarah had modified her organized activity by "looking" (§16) at the results of her prior explorations and generated a more mathematically sophisticated algorithm that allowed her to generalize beyond the 10 × 10 array.

« 12 » Let me explore a little more the question of what made Sarah's inductive, historical and recursive process of problem solving increasingly more sophisticated with each step. Totaro and Ninno's recursion of observation, *Obs Obs*(*Obs*), *Obs*(*Obs*(*Obs*)), …, indicates a "process of continuous modification of the structure" (§39), while preserving its organization. In the case of human problem solving, although solvers always produce results in successive steps (§39), not all sequences of results lead to the successful solution of problems. Sarah's recursion of problem-solving actions produced mathematically sophisticated relationships and helped her made good progress toward a successful solution. According to the authors, in a goal-directed activity such as problem solving, does the ordinal and relational nature of successive steps of a biological recursion always necessitate each step to be a better resolution of the perturbation (problem) than the steps prior to it? Q1 In other words, what would continuous modification of a structure through a recursive process *Obs*, *Obs*(*Obs*), *Obs*(*Obs*(*Obs*)), …, mean if a solver is not achieving their goal of solving the given problem? How are recursions of observation that lead to successful resolutions of problems different from those that do not?

References

Becker J. P. & Shimada S. (1997) The open-ended approach: A new proposal for teaching mathematics. National Council of Teachers of Mathematics, Reston VA.

Burger W. F. & Shaughnessy J. M. (1986) Characterizing the van Hiele levels of development in geometry. Journal for Research in Mathematics Education 17(1): 31–48.

Cifarelli V. V. (1998) The development of mental representations as a problem-solving activity. The Journal of Mathematical Behaviour 17(2): 239–264.

Cifarelli V. V. & Sevim V. (2015) Problem posing as reformulation and sense-making within problem solving. In: Singer F. M. Ellerton N. F. & Cai J. (eds.) Mathematical problem posing: From research to effective practice. Springer, New York: 177–194.

Cobb P. (1985) Two children's anticipations, beliefs, and motivations. Educational Studies in Mathematics 16(2): 111–126.

Freudenthal H. (1991) Revisiting mathematics education, China lectures. Kluwer Academic Publishers, Dordrecht.

Glasersfeld E. von (1991) Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget's approach. In: Steffe L. P. (ed.) Epistemological foundations of mathematical experience. Springer-Verlag, New York: 45–67. ▸ https://cepa.info/1418

Kilpatrick J. (1986) Reflection and recursion. In: Carss M. (ed.) Proceedings of the Fifth International Congress on Mathematical Education. Birkhäuser, Boston: 7–29.

Lave J. (2016) Word problems: A microcosm of theories of learning. In: Light P. & Butterworth G. (eds.) Context and cognition. Routledge, New York: 74–92.

Maturana H. R. & Varela F. J. (1980) Autopoiesis and cognition: The realization of the living. Reidel, Dordrecht.

Maturana H. R. & Varela F. J. (1987) The tree of knowledge: The biological roots of human understanding. Shambhala, Boston.

Steffe L. P. (1988) Cognitive conflict in learning mathematics. In: Wilson P. S. (ed.) The role of inconsistent ideas in learning mathematics. Symposium for the Special Interest Group for Research in Mathematics Education of the Annual Conference of the American Educational Research Association. University of Georgia, Athens: 47–50.

Steffe L. P. (1994) Children's multiplying schemes. In: Harel G. & Confrey J. (eds.) The development of multiplicative reasoning in the learning of mathematics. SUNY Press, New York: 3–39. ▸ https://cepa.info/2101

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Continuous versus Discrete: A Transitioning Dynamics for a Constructed Dichotomy?

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> Abstract • I point out that seamless transitioning between the discrete and the continuous is a characteristic trait of mathematics and that the continuum and the discrete are our constructions as living systems. Moreover, Peano's very basic recursion seems to fall short of providing a mathematical model for "biological recursion" (autopoiesis, self-reference, structural coupling), which involves a complex never-ending action–perception helix, for which the necessary mathematics is still a "work in progress."

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« 1 » Paolo Totaro and Domenico Ninno address, in their target article, the discrete – continuum quandary in human cognition, particularly in the mathematical context, and try to apprehend it in terms of the biology of cognition à la Humberto Maturana and Francisco Varela (1980), a most commendable endeavor indeed. This is an admittedly complex issue (Bell 2022).

« 2 » It should be noted that, in mathematics, the discrete/continuous dichotomy is not so clear-cut. Indeed, a relevant aphorism for mathematics could be: "mathematics is the art of seamlessly transitioning between the discrete and the continuous." This is more visible in some mathematical schools than in others: for instance, in the old Soviet school of Israel Gelfand, Yuri Manin, Igor Shafarevich, Alexander Kirillov and Vladimir Arnold (see Kirillov 1976, for an example) but also in the French Bourbaki school (Corry 2004). The *principle of continuous induction*, which many of us learnt in an undergraduate calculus course, is a nice example of a *continuous* analogue of the classical principle of (discrete) *mathematical induction* (Hamkins 2020: 2.4), which Totaro and Ninno deal with at length. Also, *non-standard analysis* (Hamkins 2020: 2.7) provides a smooth