which focus on large-scale challenges (Trna & Trnova 2015), such as sustainability, global health, and urban planning, where traditional pedagogical paradigms are becoming obsolete (Dutta 2020).

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Unraveling a Royal Road to Math Education

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> Abstract • In their target article Valdés-Zorrilla et al. postulate that the typical ways of teaching mathematics cause serious mental-health problems in a significant number of our students. They alert us that this would be "the most critical problem in mathematics education." They propose a solution based on metaphorizing along with embodied and enactive strategies, and exemplify them in work on probability. I argue that the great challenge is how to get the educational system to adopt the proposed solution. This solution may also help another critical mathematics education problem that is coming our way, i.e., trust.

« 1 » In my commentary on Amaranta Valdés-Zorrilla et al.'s target article, I discuss the following four aspects: (a) I agree that the mental-health problem is critical and that the solution proposed in the article is appropriate. (b) I argue that this problem goes beyond mathematics and suggest three possible mechanisms that together would be the cause of cognitive bullying. (c) I stress that the proposed solution has a history of several millennia, but has not made it to the classroom yet. We need to understand the nature of the barriers that hinder its adoption. (d) I argue that the most critical problem in mathematics education now is trust. It is at the base of our social life, but now AI is pushing us into completely unknown territory.

« 2 » Valdés-Zorrilla et al. warn that "cognitive bullying" is "the most critical problem in mathematics education" (§1). The mental health of students is currently a critical problem, which has worsened significantly since 2012. As Jonathan Haidt and Sean Twenge[1] report, both suicides and mental-disorder episodes have grown markedly. One possible cause is academic performance. However, as Haidt and Twenge point out, the incidence is very low, and the number of cases has not increased. According to the Trends in International Mathematics and Science Study (Rutkowski & Rutkowski 2018), which provides trend data on the mathematics and science achievement of students around the world, bullying among fourth-graders correlates negatively with math performance. However, in countries with better results and more intensive traditional teaching,[2] there is less bullying. History can give us some further clues. For more than 4,000 years, there has been evidence of child mistreatment in education. Sumerian tablets describe "rowdy schoolrooms, full of boisterous, disobedient pupils who were frequently caned for misbehavior" (McGuinness 1997: 44). One excerpt from a tablet says "my teacher reading my tablet said: there is something missing, and caned me." And the student ends up lamenting, "I began to hate the scribal art and to neglect the scribal art" (ibid). Student abuse seems to be an age-old problem and not just in mathematics education.

« 3 » What could be the cause of cognitive bullying? I suggest three possible explanations. First, we have to differentiate biologically primary knowledge, such as walking or talking, from biologically secondary knowledge, which is the knowledge taught at school (Geary 2007). The ability to learn biologically primary knowledge is innate. However, learning biologically secondary knowledge (such as the mathematics prescribed in the official curricula of the countries) requires many years of schooling with countless hours of dedicated teacher guidance. It provokes important changes in neuronal areas. Stanislas Dehaene and colleagues (2010: 1363) report that, for example, literacy "boosts the organization of visual cortices," and that existing cortical mechanisms for visual recognition are recycled for reading and writing. In Dehaene et al. (2015: 241) they report that literacy causes "the thickening of the splenium and/

1| See their unpublished manuscript "Adolescent mood disorders since 2010: A collaborative review," retrieved on 28 February 2023 from https://tinyurl.com/TeenMentalHealthReview

2| See OECD report "Creating effective teaching and learning environments: First results from TALIS," 2009. https://www.oecd.org/education/school/43023606.pdf or the isthmus of the corpus callosum." Lev Vygotsky (1978) had already claimed that the teaching of written language requires years of "artificial training."

« 4 » Second, the process of learning biologically secondary knowledge uses unique predispositions of humans for social learning. Unlike other animals, humans over-imitate (Tomasello 2016). When shown procedures to achieve a goal, which contain clearly irrelevant steps, chimpanzees skip over them while pre-schoolers imitate them completely. We over-imitate and learn most of the secondary biological knowledge from what others tell us. This is particularly critical in abstract knowledge, which is not directly observable, like germs and oxygen (Harris 2012). According to Paul Harris (2012), the appropriate cultural metaphor is not the child as a scientist, who explores and inquires, but the child as an anthropologist who seeks to imitate and assimilate into an unknown tribe.

« 5 » Third, imitation and recombination are the building blocks of cultural evolution (Henrich 2016). In this way, we invent new knowledge but do not necessarily obtain understanding. For example, corn not properly processed causes the deadly disease, pellagra. Unlike the Native Americans, Europeans suffered from it and it took centuries to understand what caused it. They had not copied the entire original process of corn treatment. However, the Native Americans did not know the reason either. They just followed it ritually. These three mechanisms would be causing cognitive bullying. We need to learn an overcrowded curriculum of biologically secondary knowledge. We do it using over-imitation and without much comprehension. These mechanisms evolved for learning hunter-gatherer skills. They are not well adapted for learning the current curriculum, with its demands of analytical comprehension, or for doing so efficiently under pressure from teachers and high-stakes assessments (Chaudhary & Swanepoely 2023).

« 6 » Cognitive bullying poses a genuine educational challenge. I fully agree with the solution Valdés-Zorrilla et al. propose in the target article: metaphorizing along with embodied and enactive strategies, which altogether contribute to a deeper understanding. It has a history of several millennia. In a letter to Eratosthenes, Archimedes described his mechanical method that uses levers to balance figures in order to conjecture mathematical properties: "This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, […]" (Heath 1912). Vygotsky (1978) highlights how physical artifacts play the role of mediators of human thought. In algebra, the use of scales and boxes allows low-performing students to solve mentally first-degree symbolic equations with the same ease as high-performing students who solve them in the traditional way (Araya et al. 2010). In the 17th century, Comenius proposed the "testimony of the senses" in science education (Rabecq 1957). Many others have proposed it, but today it is still not common in classrooms (Cuban 2013). We need to understand the nature of the barriers that hinder its adoption. Solutions like the one proposed by Valdés-Zorrilla et al. are very demanding in time, energy and skills. This leads to a lot of resistance from teachers. One way to manage these demands of highly active lessons is through technology. For example, if all 40 students respond simultaneously in writing, an AI classifier can alert the teacher to incoherent responses without delay and help her better assess student learning (Urutia & Araya 2022).

« 7 » In their exemplification of the concept of probability, Valdés-Zorrilla et al. (§24) use random walks travelled by frogs. This is a very effective and engaging strategy because the random walks are performed by agents who are concrete characters that students, as early as in the first grade, can enactively use to address the mathematical modeling of a wide spectrum of probabilistic phenomena that are core to science (Araya 2022).

« 8 » Events are often abstract, in the enactive sense that they are not "whole objects," i.e., "bounded, cohesive entities that move as a unit, independently of other surfaces" (Brase, Cosmides & Tooby 1998: 8), and that even for an infant they are easy to count. However, translating abstract events into "whole" objects facilitates the solution of probability problems. The solution to the urn problem proposed by Valdés-Zorrilla et al. of leaving the balls aligned is concise, elegant and precise. Still, I would like to suggest two alternatives. First, let us look at the problem qualitatively. Before the first random draw of a ball, there are fewer red than blue balls, which means that after the draw the proportion of reds remaining in the urn is slightly more likely to increase than to decrease. However, the size of the eventual increased proportion is slightly smaller than the size of the eventual decreased proportion. Combining both effects, the proportion remains the same. We could calculate both effects numerically and verify that they balance. Thus, after randomly removing the first ball we are back at the beginning, but this time with only four balls in the urn. The second alternative I can think of has to do with natural frequencies (Gigerenzer et al. 1987). Leda Cosmides and John Tooby (1996) argue that we have cognitive mechanisms adapted to frequentist reasoning. Single events easily confuse as. In the case of the urn, I suggest imagining a hundred urns, and in parallel drawing a random ball from each urn. One would expect that, on average, in 40 urns a red ball would come out and in 60 urns a blue one. Then among the 100 urns we would have an average of 40 urns in which one out of every four balls is red, and 60 urns in which two out of four balls are red. That is, 40 red on the one hand, and 120 red on the other. On average 160 of the 400 balls will be red, that is, two red for every five balls. We are then as at the beginning, but with only four balls in the urn. Can we assess how easily understandable and generalizable these strategies are? Q1

« 9 » Valdés-Zorrilla et al.'s solution is about metaphorizing, embodiment and enaction. Including dance and choreography makes it an effective educational strategy. There are parallels with religion, which, like mathematics, works with several notions that are not directly observable. They are abstractions that require sophisticated cognition of at least five degrees in recursion: each person can imagine what another thinks about what a third person thinks and so on until arriving at what a fifth person thinks (Dunbar 2022). In ancestral immersive religions and many religions currently practiced, song and dance often play a central role. The embodiment of abstractions such as the "holy spirit" and its enactive practices encompass jumping, dancing, shouting, prophesying, speaking in tongues, anointing, laying on of hands, and praying, achieving the extremely active and synchronous participation of all. As shown by Robin Dunbar (2022), the synchrony triggers the secretion of endorphins that increase bonding and a sense of belonging. In addition, touching one another, at a moderate speed, activates C-tactile neurons, which also generate the secretion of endorphins and increase cohesion. These embodied and enactive strategies help engagement and understanding, and can also be an effective inoculation against "cognitive bullying."

« 10 » Another interesting aspect is Valdés-Zorrilla et al.'s call for using micro-phenomenological interviews in research. Let me take this a step further. Suppose we connect to a super mind-reading and mind-writing[3] device such as an extended version of the experience machine proposed by Robert Nozick (1974). It would be a machine connected to all our senses and muscles, and that allowed us to walk through didactically predesigned spaces. It would scan our brain, decode its signals using AI, and thus access our thoughts, even those we are not consciously aware of. It would be an ideal micro-phenomenological machine. It could perform machine-to-brain intersubjective nonverbal communications (Shore 2021), directly interrogating our brain with a greater temporal and spatial resolution than micro-phenomenological interviews can achieve. If a new emerging problem-solving strategy were not yet conscious, then the reading device could detect it and immediately interview (similar to the micro-phenomenological interviews) the student to bring it to consciousness. Would such a device give us less traumatic learning and a deeper understanding? Q2

« 11 » The great challenge is figuring out how to get the educational system to adopt the proposed solution. This solution may also help another critical education problem that is coming our way. This new problem is even more pressing than "cognitive bullying." This is the problem of trust (Araya 2021). It encompasses all students, including the gifted ones. Trust is at the base of our social life, but now AI is pushing us into completely unknown territory. A growing number of artificially intelligent agents, such as ChatGPT and other large language models, are beginning to advise us and take control of our inner and social life. However, they may not always give the same advice. They could even lie[4] and might respond to the interests of third parties, not ours. Evolved face-to-face and reputation-based trust-building mechanisms are maladapted to this new environment. This unexpected and disconcerting irruption may provoke mental-health problems never seen before. Since these technologies will be ubiquitous, citizens will need to understand the basic principles of how these apps work and preferably have some understanding of the mathematical models[5] and algorithms they use. This is the enormous challenge that we now have to address in mathematics education.

3| Mind reading refers to accessing mental processes, and mind writing is about stimulating the brain (Roelfsema, Denys & Klink 2018).

4| Even if AI agents do not have intentions, Guglielmo Papagni and Sabine Koeszegi (2021) argue that the attribution of intentions could be a good strategy to understand and predict the behavior of these agents.

5| In Araya (2022) I provide some examples of first graders getting introduced to mathematical modeling.

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Is it as Simple as "E-Teaching Good" Versus "Standardised Teaching Bad"?

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> Abstract • While sympathetic to much of what the authors of the target article write, I claim that there is a binary in their thinking, i.e., the one in the title of my commentary. I argue that such binaries obscure some groups and, in this case, one thing relatively hidden is the role of the teacher, in the lessons described. An alternative to positing an opposition between teaching approaches would be to apply the principles of "E-teaching" to working with teachers, as well as students.

« 1 » I would like to open by commending Amaranta Valdés-Zorrilla et al. on their desire to make the teaching and learning of mathematics a more humane and enriching experience for students across the world. I share their sense of the importance and urgency of such an endeavour. I also appreciate the authors' introduction of the idea of "metaphorising" as central to learning mathematics (i.e., viewing the creation of metaphors as the mechanism by which concepts are enacted), and their broader exposition and elaboration of enactivist principles, to inform their work in classrooms. I have engaged in similar work over many years, convinced that enactivism has much to offer within education.

« 2 » Furthermore, I value greatly the descriptions of the random-walk task and its variants. When I was teaching in a secondary school in the UK, the department I was in used a similar task (which we had not linked to random walks) where a counter moved left or right down a grid and questions arose as to the likelihood of its ending up at other points in the grid. In our school, we had a practice of working on extended tasks, which students would do over a number of weeks and with the explicit intention that students would pose their own questions and follow their own lines of enquiry. So, in this task, students would be encouraged to vary the size of the grid and also the probability of moving left/right, and to look for patterns, generalisations and explanations. I found myself convinced by the power of the teaching, as described, perhaps helped by having experienced positive student reaction to some similar tasks.

« 3 » The links made between mathematics, movement and art are compelling ones to me and the description of classroom tasks will, I hope, inspire others to try some similar things. The way the authors call out the potential for mathematics teaching to be tantamount to cognitive bullying will surely give anyone reading this, who is a teacher at whatever level, cause for reflection on their own practices.

« 4 » The key point I want to make, by way of provocation, is to register a wariness about "binaries" in our thinking, especially when those binaries set up one side as "good" and the other as "bad." This wariness comes from a reflection that binaries of good/bad have, historically, been used to marginalise groups of people, across the world. Binary thinking tends to oscillate and lead to argument and conflict, since it pur-