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The Ambition of Absolute Agreement in Mathematics, and Deviations from It

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> Abstract • I claim that mathematics is constructed with the ambition of allowing for absolute agreement. Dealing with students' thinking that deviates from the supposedly absolute agreement is a key problem for mathematics teaching. Some sources for such deviant thinking include the relationship between mathematics and modelled reality, and the "imaginary" nature of probability.

« 1 » In their target article, Amaranta Valdés-Zorrilla, Daniela Díaz-Rojas, Leslie Jiménez and Jorge Soto-Andrade present some very inspiring thoughts and ideas. I will comment on a few aspects that I believe are worthwhile considering in connection with the article, namely (a) the handling of mathematical "errors" and its connection to "cognitive abuse" in mathematics teaching, (b) the relationship between mathematics and modelled reality, and (c) the problematic nature of probability.

Mathematical "errors"

« 2 » Valdés-Zorrilla et al. use the term "cognitive abuse" for much of mathematics teaching. I certainly agree that "maths anxiety" is a very frequent phenomenon, and can often be seen as the result of problematic mathematics teaching.

« 3 » However, the authors do not discuss what is special about mathematics compared to other subjects that makes mathematics teaching particularly susceptible to anxiety and "cognitive abuse." This might have a lot to do with the widespread belief that mathematics is the realm of the most certain and objective knowledge. Paul Ernest (1998) calls this "absolutism." For example, in 1929, David Hilbert portrayed "the generally held opinion about mathematics and mathematical thought" in this way:

> "The mathematical truths are absolutely certain, for they are proved on the basis of definitions through infallible inferences. Therefore they must also be correct everywhere in reality." (Hilbert 1998: 22)

« 4 » Furthermore, there is also the belief that the "truths of mathematics and logic appear to everyone to be necessary and certain" (Ayer 1952: 72). Everyone is expected to understand their necessity and certainty! Lack of understanding of mathematics is often interpreted as a sign of a lack of intelligence, by teachers and society as well as by students themselves, and can have a more negative effect on self-esteem and confidence than a potential lack of knowledge in other domains.

« 5 » Many teachers of mathematics would probably agree that the free development of the mathematical insight of the learner based on given or even self-chosen mathematical problems would be a "royal road" to the learning of mathematics, rather than coercion and rote learning. However, there are practical obstacles such as time pressure, a given syllabus, and the experience that many students do not arrive at the expected conclusions on their own, i.e., those seen as "true" according to a consensus of the mathematical community. Mathematics teaching that encourages students to think on their own needs to deal with ways of thinking and problem solving that deviate from this consensus. This aspect is neglected by Valdés-Zorrilla et al. In their examples, supposedly "wrong" conclusions are only drawn by learners before being exposed to their approach, and the potential of their approach to lead to supposedly "correct" mathematical insight is emphasized.

« 6 » My view, elaborated in Hennig (2010), is that the ambition that drives the construction of mathematics as we know it is the creation of a communicative system within which absolute agreement is possible. Empirically, the mathematical community has been quite successful at this, at least in the sense that, according to my experience, most people either agree with the parts of the established body of mathematical knowledge that they know, or declare themselves incompetent rather than doubting this knowledge.[1] For the ambition to be successful, there should be no need for the agreement to be externally enforced. Rather, in order to become established mathematical knowledge, claims are supposedly constructed in such a way that everybody with enough insight will agree automatically. However, my experience of mathematics teaching is that this does not always work, be it for lack of time or for other reasons. A mathematics teacher who perceives themselves as responsible for conveying agreed mathematical knowledge to the students will therefore be tempted to enforce agreement by authority where it does not happen otherwise. If this is the case, then the impression of "cognitive abuse" results from the dissonance between the mathematical ambition that agreement should result from free thinking and personal insight, and the authoritative act of enforcing the agreement, brushing aside existing doubts and thoughts in case the student's insight does not come about as intended by the teacher. Later, taught material is assumed and claimed to be agreed on. In mathematics, the dependence of new knowledge on previously taught knowledge is stronger than in other domains, contributing to the stability of mathematical knowledge. In turn, students who keep having doubts about mathematical knowledge are made to feel that something

1 | However, exceptions exist (Dudley 1992).

https://constructivist.info/18/2/259.valdes-zorrilla

is wrong with their thinking (even without requiring teachers to explicitly declare them to be lacking in intelligence).

« 7 » A fundamental condition of mathematics teaching is the tension between encouraging the free active and creative thinking of the students (which according to the mathematical ambition should lead to insight and agreement) and the necessity (perceived and/or often enforced by background conditions) to ultimately convey the supposedly "absolutely agreed upon" mathematical knowledge. The necessity to have absolute agreement seems constitutive for mathematics and its constructive value, so it should not be all too easily dropped. With open-ended problems and teaching concepts as presented in the target article, "correct" mathematical thinking that is in line with the consensus of the mathematical community can go far beyond straightforward solutions to simple problem tasks, but the tension remains. The teachers themselves may, in principle, be prepared to question the alleged absoluteness of mathematical knowledge. From my own teaching experience, however, I can say that well-understood mathematics does indeed look strong and stable, so that the temptation is great not to let students get away with views that run counter to it.

« 8 » The attitude of the authors of the target article seems to be to have trust in the students to arrive at something worthwhile in line with the mathematical consensus, and what is proposed is probably indeed helpful in this respect. However, they do not deal with the question of what to do if their method does not work for certain individual students. So, I wonder, how to deal with mathematical "errors" that are obvious for the teacher but are nevertheless held by students. To what extent can any deviating ideas of the students be tolerated or even appreciated? Q1

Mathematics and modelled reality

« 9 » Another important aspect is the issue of the mathematical modelling of "reality" as discussed in detail in Hennig (2010). There, I argue that mathematical formalism is a special communicative domain, and as such essentially different from the experiential reality that is modelled by it. Mathematical modelling requires abstraction, and abstraction will unavoidably clear away certain features of what is experienced as "real." Even simple mathematics such as counting, say, cows and representing them by notches, will abstract away the individual differences among cows. This may be a major issue for at least some students in mathematics education in the context of supposedly practical applications. Considering a random walk of a frog or the Rayen's Fall setup in §76, the thinking of some students may focus on details abstracted away by the probability models such as the dependence of where the frog leaps on what exactly it sees from where it is, which may depend on the time of day and the weather. Also, Rayen may become better at negotiating the wet road after a few rainy days. This may seem more important to a student than the requirement of abstraction for doing calculations that may seem inappropriate for the practical situation, and it can be a motivation for the deviation of student thoughts from the potential mathematical insight to be learned here. Being aware of this aspect can be very helpful for a teacher when faced with a seeming lack of mathematical understanding. The historically grown strong (and supposedly absolute) agreement regarding purely mathematical knowledge does not generally apply to the connection between mathematical formalism and the experiential reality to be modelled. A student's potential objections to this connection whenever reference to mathematically modelled experiential reality is used in mathematics teaching does not contradict established mathematical knowledge.

« 10 » I am not arguing against the use of examples where mathematics is applied to practical or everyday situations. Valdés-Zorrilla et al.'s encouragement of metaphorizing is welcome, and clearly the authors do not promote a naive identification of experiential reality and mathematics; the concept of metaphor already involves a perception of difference. I would, however, like to raise awareness of the tension between the use of "practical application examples" for illustrating the relevance of mathematics and connecting it to the perceptions of some students, and the danger of identifying mathematics all too uncritically with experiential reality. Some other students will be served better by acknowledging the differences.

The problematic nature of probability

« 11 » I conclude my commentary with a remark on probability, which is a relatively young concept in mathematics. Its interpretation is problematic and controversial (e.g., Hacking 1975). A major reason for this may be that probability essentially does not only refer to what we perceive as "real," but also to counterfactuals, i.e., what did not happen but could have happened (such as a single frog leaping to the right when it could have leaped to the left). The connection between probability and observation is therefore not straightforward. In §§54–60, the authors nicely speculate about metaphors that represent "hallucinated" events ("metaphorising is *hallucinating*," §21) that are imagined as material and not counterfactual, which may help a beginner to grasp the at best half-concrete nature of probability.

« 12 » Still, this problematic nature may be another opportunity for certain students' thoughts to go in a direction that is different from the mathematics they are meant to learn, as their intuition may not correspond to the standard probability model setup. Issues can be, for example, the difficulty of differentiating between probabilities apart from (near) certainty or (near) impossibility, and the distinction between stochastic dependence and independence (the latter being quite relevant for Rayen's Fall). Consistent with this, there is much research showing that deviations from established reasoning with probability are very common (e.g., Tversky & Kahnemann 1974; O'Connell 1999).

« 13 » Such potential deviations of students' reasoning from established mathematical knowledge can be better acknowledged and valued by a teacher with an open mind and a strong resolution to relate to the students' thoughts. This can help to counter "abuse" and coercion without insight, even though it will not remove the tension resulting from students who remain unconvinced of the established knowledge that is supposedly to be taught. In any case, awareness regarding potential sources of discrepancies between the learner's intuition and reasoning guided by established mathematical knowledge such as the nature of probability and the necessary differences between mathematical models and experiential reality may help the teacher to connect better to deviant student views.

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Random Walks on Structures, Autopoiesis and Meta-learning

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> Abstract • I discuss how random walks provide a metaphorical playground for mathematical conceptualization. In particular, I propose to extend the random-walk metaphor, by introducing structural changes that go beyond mere changes of state. Hence, the "steps" in the walk change not only the position the system is at, but also the way in which it walks. The latter invites learners to think in constructivist concepts in a mathematical way.

« 1 » Being a former student and teaching assistant of one of the authors of the target article, Jorge Soto-Andrade, I could not help but have beautiful memories of that time. I was fortunate to witness at first hand the usefulness of the random-walk metaphor for introducing mathematical concepts. Indeed, when I taught students who were not mathematically driven (especially those that had suffered cognitive bullying), the guided conceptual explorations of the metaphor generally provided a safe space for reflection. Mathematical concepts blossomed in a smooth and fun way, and students appreciated the difference from more traditional pedagogical styles where mathematical concepts are imposed as absolute truths.

« 2 » However, for students pursuing a specialization in mathematics (e.g., mathematics or theoretical physics), the situation can be completely different. The metaphor can become challenging for minds that are "too trained" in traditional mathematical setups. Often, mathematics students tend to reject intuition and approach learning as a procedure where sophisticated symbolic manipulation and analytic reflection are assumed to be the best way to learn. For many of them, utilizing random processes as a way of imagining situations becomes an uncomfortable experience, as they believe that mathematical concepts can be "easily understood" (clearly, I am suspicious of their notion of understanding here) when presented as logical propositions, making other "pictorial" and "case-based" ways of introducing them seem inefficient and unnecessary.

« 3 » The attitude of this group of students reflects the tendency of mathematics education to instruct students to prefer symbolic manipulation and analytic reasoning over intuition or associative thinking, and consider the former as superior ways of thinking over the latter. The latter reflects a generally unconscious, but aggressive, strategy aiming at the self-preservation of the privileged place of mathematics in the educational world, and thus the historical reason for cognitive bullying in mathematics. Therefore, "mathematics education" could be understood as a historical-social complex that has self-preserved over centuries, creating boundaries and power asymmetries in the educational world.

« 4 » These general concerns aside, I completely agree with Amaranta Valdés-Zorrilla et al. that the metaphoric situations expressed by random walks are useful to introduce mathematical concepts in a fresh and emotionally safe way. In particular, it should complement the traditional skills that are developed among specialists in mathematics. Still, I would like to propose that a more elaborated form of random walks, where the steps of the walk change the walking possibilities, might lead to surprisingly interesting reasoning in students.

« 5 » In general, a random walk can be well metaphorized by the movement of a frog over a specific geometry. The locations visited by the frog (and potential replications or divisions of it along the way) explain notions of stochasticity and invite students to frame problem-solving strategies as explained in the article (§23). However, in this metaphor, frogs are immutable objects: They can only move or reproduce but they cannot change into something other than a frog. This prevents some deep aspects of the notion of "identity" from coming into play in the reasoning evoked by the metaphor.

« 6 » When the random walk extends in such a way that the frog can acquire or lose features along the course of its movement, the situation becomes more interesting. Let us, for example, consider a simple case where the frog, which is constantly moving among different spots, begins as a baby frog and grows after some time (movements) to the point that