gagement. Educational Psychological Review 27: 391–404. ▸ https://cepa.info/6847
**Gandell R. & Maheux J.-F. (2019)** Problematizing: The lived journey of a group of students doing mathematics. Constructivist Foundations 15(1): 50–60. ▸ https://constructivist.info/15/1/050
**Gibbs R. (ed.) (2008)** The Cambridge handbook of metaphor and thought. Cambridge University Press, Cambridge UK.
**Hohr H. (2013)** The concept of experience by John Dewey revisited: Conceiving, feeling and "enliving." Studies in Philosophy and Education 32: 25–38.
**Lakoff G. & Núñez R. (2000)** Where mathematics comes from. Basic Books, New York NY.
**Proulx J. & Maheux J.-F. (2017)** From problem-solving to problem-posing, and from strategies to laying down a path in solving – Taking on Varela's ideas for mathematics education research. Constructivist Foundations 13(1): 160–167. ▸ https://constructivist.info/13/1/160
**Sevim V. & Cifarelli V. V. (2013)** The co-evolution of problem posing and problem solving in the course of Sarah's on-going solution activity. In: Lindmeier A. M. & Heinze A. (eds.) Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education. Volume 5. PME, Kiel: 165.
**Sfard A. (1991)** On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics 22(1): 1–36.
**Sfard A. (1994)** Reification as the birth of metaphor. For the Learning of Mathematics 141: 44–54.
**Varela F. J. (1992)** Autopoiesis and a biology of intentionality. In: McMullin B. (ed.) Proceedings of the workshop Autopoiesis and Perception. Dublin City University, Dublin: 4–14. ▸ https://cepa.info/1274

**Volkan Sevim** is Associate Professor of Mathematics in the Department of Computer Science and Mathematics at the University of South Carolina Beaufort. He received his PhD in Curriculum and Instruction with a specialization in Mathematics Education from the University of North Carolina at Charlotte. His current research examines learning processes involved in the co-evolution of problem solving and problem posing. Volkan has presented his research at NCTM, AERA, PME, and PME-NA, and has taught upper-level mathematics courses at a public high school in Charlotte, NC, for five years. In the last ten years, he has been teaching mathematics and mathematics education courses at university level.

**Funding:** The author has not received any external funding while writing the manuscript.
**Competing interests:** The author declares that he has no competing interests.

Received: 15 June 2021
Revised: 19 June 2021
Accepted: 21 June 2021

## Authors' Response

# Fathoming the Enactive Metaphorizing Elephant in the Dark…

Daniela Díaz-Rojas
University of Oxford, UK • daniela.diazrojas/at/education.ox.ac.uk

Jorge Soto-Andrade
Universidad de Chile, Chile
sotoandrade/at/uchile.cl

Ronnie Videla-Reyes
Universidad de la Serena, Chile
rvidela/at/userena.cl

**> Abstract** • We offer a response to three themes arising from the commentators' inquiries and critiques: (a) The epistemological compatibility of enactivism and conceptual metaphor theory; (b) the way enactive metaphorization works in the teaching and learning of mathematics, particularly in problem-posing and problem-solving activities; and (c) the nature of mathematical abstraction and its relation with enactive metaphorizing.

« 1 » We appreciate the contributions and thoughtful insights brought up by the commentators, which suggest ways to refine and broaden the research reported on in our target article. We address below the main three themes that we discern in the commentaries on our target article.

### Are enactivism and conceptual metaphor theory epistemologically compatible?

« 2 » We thank **Dor Abrahamson** for his insightful remarks on the epistemological incompatibility between conceptual metaphor theory and enactivism (§3) and his proposal of a compatible alternative metaphorizing of metaphor as a *constraint* that individuals imaginatively project onto their perceptuomotor attempts to engage the environment so as to enact goal movements that achieve task performance (§4). We feel, nevertheless, that epistemological incompatibilities arise because we are fathoming the same "elephant in the dark" from different sides, as the well-known Eastern story goes. Indeed, we metaphorize metaphorizing in various ways, which are triggered by different research contexts and related activities. Challenging tasks proposed to the students play a key role in **Abrahamson**'s activities. In our case, as radical enactivists, we tend, in general, to just set up a situation, with no explicit task proposed, leaving room for students' problem posing. Unfortunately, this is not very visible in the examples provided in the target article. There, we asked precise questions because – especially in the line-crossing problem – we wanted to investigate the spontaneous affective-cognitive reactions of the students to the problem. In other words, we wanted to find out whether disliking the problem triggered idiosyncratic metaphorizing. We fully agree, though, with **Abrahamson** (§9), in his appreciation of the fundamental role of idiosyncratic metaphorizing in the development of mathematical thinking and his leitmotiv: "learning is moving in new ways" (**Abrahamson** & Sánchez-García 2016). 

« 3 » We wish to point out, also, that the metaphor of "constraint" puzzles us. Contrary to **Abrahamson** (§4), we see constraints in situations that trigger question asking and problem posing from the students, which in turn foster the emergence of idiosyncratic metaphorization in them. Also, we do not consider metaphorizing a *tool* to be used, aiming at task performance. Rather, it is a *means* for randomly exploring problematic situations, and thus a prerequisite for learning.

« 4 » In our article we probably overemphasized the "arrow metaphor" for metaphorizing when, in §22, we metaphorically characterized metaphors as inference-preserving *mappings* from one inferential domain to another. This is clearly an enticing metaphor for metaphorizing. However, as Abrahamson (§3) points out, this rather top-down theoretical framework hides the circumstances of the emergence of idiosyncratic metaphorizing in our subjects, i.e., the characterization of the conditions that foster, if not trigger, hinder, or even thwart this emergence. That is to say, the investigation of metaphorizing as a first-person experience is not addressed by conceptual metaphor theory.

« 5 » Moreover, two caveats are in order here, which are pertinent to Abrahamson's remarks. First, as we have remarked elsewhere (Soto-Andrade 2018), following Anna Sfard (1997), metaphorizing may be understood as a circular autopoietic *process* (Maturana & Varela 1980) rather than as a unidirectional source-on-target mapping. So, a more appropriate metaphor than an arrow would be a dynamic cycle, sometimes depicted as an Ouroboros, serving as a metaphor of circularity and self-reference (Soto-Andrade et al. 2011). Similarly, in Raymond Gibbs's (2011) systems theoretical approach, metaphors appear – metaphorically – as basins of attraction in the space of all possible "cognitive states" of an individual or a community. The cognitive dynamics of a subject is visualized as a trajectory in this cognitive space, which "dances" among several attracting basins.

« 6 » We also notice that there is a natural circular intertwining between metaphorizing and enacting. On the one hand, we *metaphorize* to describe enaction, as Francisco Varela did, referring to Machado's verse and Escher's lithography (§16). On the other hand, when we are enmeshed in a problematic situation that matters, our metaphorizing often emerges *enactively*.

« 7 » The second caveat is that from the vantage point of conceptual metaphor theory and ours, metaphorizing is just "the tip of the iceberg" in the cognitive domain. The rest of the iceberg, or the elephant, remains to be explored.

« 8 » Regarding enactive metaphorizing in the sense of Shaun Gallagher, as criticized by Abrahamson, we wish to point out that our eighth-graders, in §47, do not engage in "dictated" movements (§5). Rather, they just get a bunch of yarn threads to figure out how to use it to connect with one another, metaphorizing handshakes in this way. They do this "in disorder," moving spontaneously, non-systematically, in general.

« 9 » As humans, we unavoidably metaphorize, either overtly or covertly. The questions we address are: *How* do we metaphorize? *How* do we move from one metaphor to another? And, more generally, *how* do we move among a constellation of metaphors? See §§31–35 below, for specific examples.

## What are the practical aspects of enactive metaphorization in mathematics education?

« 10 » In this section, we look at the practical details of our work. These are also a major theme in Victor Cifarelli's commentary. Let us first focus on his four questions. His Q1 deals with our formulations for the handshake problem. Since, prior to receiving the yarn from us, the students had not made much progress yet, the groups had not made a clear choice between the one-by-one and the all-inclusive formulation. Some students in each group, though, began by *drawing* (not *enacting*) the handshaking, in the case of three or four friends. Most students, however, looked rather perplexed.

« 11 » Regarding Cifarelli's Q2 about when metaphorizing first appeared in the students' problem solving, our students did not spontaneously metaphorize the handshakes before we handed them the yarn threads. Having the yarn in their hands seemed, however, to have been a significant prompt to trigger the metaphorization of handshakes as connecting threads.

« 12 » In response to Cifarelli's Q3 regarding the roles of the students' acts of metaphorizing in their generalization, when we gave yarn to our students, besides suggesting a metaphorizing for the handshakes, we forced a collaborative tackling of the problem: they quickly began to string the yarn between one another, in disorder, not sequentially. Two or three groups quickly arrived at the conclusion that the yarn could embody handshakes and the other groups followed suit. We did not suggest to them to do so, we just handed them the yarn, as in an a-didactic situation (Brousseau & Warfield 2014). As hypothesized by Cifarelli in §5, we fully agree with Adalira Sáenz-Ludlow's (2004) findings that students' use of metaphors during collaborative problem solving is most powerful when the metaphors are self-generated through the collective reflections of the group, with a view to making sense of and organizing their subsequent problem solving.

« 13 » In §§6ff Cifarelli ponders about extensions to the handshake problem. We conjecture that our students would have no problem in stringing yarn this time *only* to their non-adjacent neighbors. What interests us here is, however, the metaphorical transition from the crossing-lines problem to the handshake problem, as well as exploring the yarn-thread approach as – in the sense of Abrahamson (§4) – constrained embodiment of the pencil-and-paper approach to the geometric diagonal counting problem.

« 14 » Regarding Cifarelli's Q4 on the interconnections between metaphorizations and re-presentations, we exemplify our answer in the case of his students: they chose a way of enacting the diagonal counting by walking sequentially around the polygon; they *could* have enactively metaphorized diagonal drawing to another vertex as thread stringing to another student. Then, eventually, they sat down and re-presented what they had enacted to write down the sum that gave the mathematically correct answer. Based on other examples, like the metaphorizing of a frog's random walk as a splitting process, we would agree that re-presentation (which we have not emphasized that much) intervenes after the students have enactively metaphorized their approach to a problem, when they sit down to "go symbolic" in a meaningful way, not just calculating blindly.

« 15 » Signe Kastberg's Q1 addresses the important aspect of teachers' professional development. Alluding to our somewhat elliptical closing §81, she wonders how one might create situations for teachers to explore ways of supporting enactive metaphorizing. In our case, we usually work with teachers in small random work groups, with subsequent plenary discussions. In these discussions, the teachers share their own (often incipient) enactions and metaphorizings arising in problem-posing and problem-solving contexts, and thus learn from one another. In Chile, we have noticed that primary teachers are usually more creative in this regard than secondary teachers.

« 16 » We fully agree with Kastberg's (§3) criticism of the "prevailing didactic contract," as it often falls short of supporting the enactive metaphorizing of learners. A substantial reshaping of this contract will be required, if we believe in the importance of legitimizing idiosyncratic metaphorizing in classroom discourse (Abrahamson §9).

« 17 » In §6, Kastberg also points out that what teachers do goes beyond making room for the students "to unfold their natural metaphorizing and enacting powers" and this results from the teachers' ability to *listen hermeneutically*. We appreciate Kastberg's emphasis on "teaching as listening," because to us leaving room for teachers and students to explore implicitly entails that we need to listen to their exploring while "bracketing" our own view of the situation.

« 18 » Jean-Francois Maheux (§5) evokes a first level of metaphorizing and enacting regarding our Greek geometry problem (§54). Surely, for us and our undergraduates, drawing lines and points is already automatic, but not so for primary school children. Greek plane geometry is still a sort of esoteric game or ritual for most of them, with gratuitous rules (anecdotal observations).

« 19 » Regarding Maheux's Q1 about the aspect of familiarity in enactive metaphorizing, let us consider, by way of example, the enaction of the well-known staircase-climbing problem: In how many ways can one climb a staircase, assuming one can climb one or two steps at a time? It may be seen as a metaphorical approach to Fibonacci numbers. Most students are able to *see* the Fibonacci recursion formula when they *enact* the climbing, because the moment they lift one foot to begin, they must decide whether to climb one or two stairs at once. The usual approach, though, is to calculate by hand, or rather "by foot," the total number of ways of climbing staircases with 1, 2, 3, 4, 5… stairs, look at the data, notice a pattern, and try to prove it in general, usually by induction. This is an example of enacting supported by the everyday experience of students.

« 20 » Regarding the examples in the target article, Maheux (§9) points out that hand shaking is a process that takes place over time, while line crossing is instantaneous. We agree, unless one enacts drawing the lines slowly one by one and zooms in on their sequential crossing of the other lines. Interestingly, when one metaphorizes handshaking by yarn threads, the original lines become nodes (the friends) and the crossing points become connecting strings, like the classical duality between points and lines in projective geometry!

« 21 » Attending to Maheux's Q2 on what constitutes good metaphors or analogies, we claim that "mathematics is the art of metaphorical navigation." Indeed, all metaphors "are not born equal" (Abrahamson §8) and metaphorical exploration is part of the students' learning process. Remarkably, *sharing* their incipient (good or bad) metaphorical approaches to challenging problems positively impacts on their learning. This sharing is especially meaningful for them when carried out in small random groups, not just in a plenary setting. They learn from one another's errors, in particular from "inadequate" metaphorizings, e.g., those that do not enable them to solve their problem.

« 22 » Let us now zoom in on how enactive metaphorizing is brought forth (Maheux's Q3). Here, more research is needed based, e.g., on a micro-phenomenological approach, in the sense of Claire Petitmengin (2006). In any case, as radical enactivists, we intend to be *minimalist* by presenting open-ended a-didactic situations (Brousseau & Warfield 2014), which are barely *situational seeds* offering, at most, implicit prompts. This is no easy task, as it requires the gesture of *phenomenological reduction* (Varela 1996) from the teacher or facilitator, that we described as "bracketing" in our comment on Kastberg's (§4) hermeneutic listening.

« 23 » Volkan Sevim (§4) discusses how metaphorizing can be considered as problem posing, and thus as sense-making. He suggests that "problem posing|solving" can be seen as enactive metaphorizing, in that the "posing part," triggered by the given prompt, is the metaphorizing part, and the "solving part" is the enaction of the created metaphors. This makes sense to us in general terms, because we enact a problem when we pose a problem motivated by a constraining situation. In our line-crossing example, though, the problem was *posed* to the students. This setting was not as radically enactivist as just presenting a *situational seed* with no explicit problem (Díaz-Rojas & Soto-Andrade 2017), as Kastberg (§§5f) points out.

« 24 » In our crossing-lines example, student metaphorizing is involved in the reformulation of the original arid geometric problem, posing a new problem instead of the given one: the *posing part*, in Sevim's sense. Then, the students metaphorize enactively to solve the new problem (the handshaking problem) by connecting themselves via yarn threads, which then they try to count in a collaborative way: the *solving part*, in Sevim's sense. This example suggests that chains of metaphorizings and enactions may arise (Sfard 1997), where the enaction of a metaphor may "feed" a subsequent metaphorization.

## What is the nature of mathematical abstraction and its relationship with enactive metaphorizing?

« 25 » Cifarelli (§15) points out that his students' problem solving was abstract in that they could generalize from their actions to find the formulas and verify algebraically their equivalence. Hence, their problem solving challenges our assertion that abstract activity "may be a dramatic cognitive impoverishment, which certainly hinders our creativity" (§80). We would contend, however, that injunctions such as "you are counting twice so you must divide by two" remain quite opaque for young students, unless they manage to embody it somehow, e.g., with the help of yarn threads.

« 26 » Deriving the $(n-3)+(n-3)+\dots$ formula (Cifarelli §10) seems to have been quite an enactive process. The student imagined drawing the diagonals stepwise from vertices in a well-chosen sequence, paying attention to avoiding repetitions. This is an iconic rather than an abstract-symbolic procedure. Checking algebraically the equality of both formulas is abstract. However, we could claim that one has an enactive proof of their equality, because one has just counted the same number of objects in two different ways, involving different actions and movements, tangible or imagined. To be thoroughly abstract would mean to ask the students to prove the equality algebraically, with no context, no embodiment, no iconic representation whatsoever.

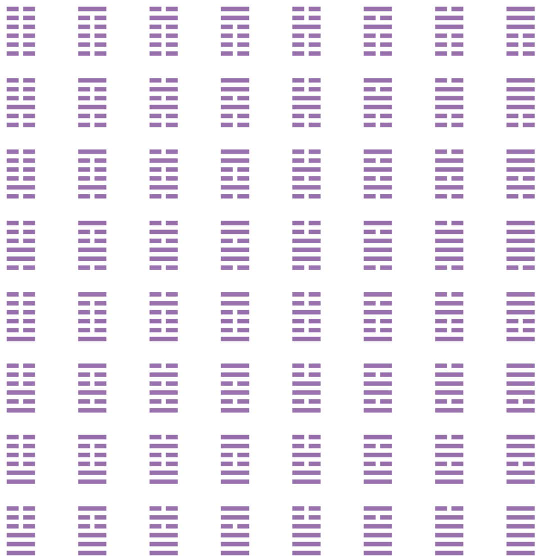

**Figure 1** • Shao Yong's 8 × 8 square.

« 27 » It is valuable to induce a conjecture from numerical evidence and then find a general argument to prove it. It may be an abstract argument, even a proof by induction (which is not especially transparent or creative, though), or an embodied or metaphorical proof. In our research, we are interested in the latter, which seem usually to be dismissed or neglected in mainstream math education. This phenomenon is more dramatic in harder problems, where students who remain on an abstract level often do not "see" why a surprising result holds, reporting that they calculate abstractly but without understanding (Soto-Andrade 2018).

« 28 » In response to **Anderson Norton & Vladislav Kokushkin**'s **Q1**, we see higher-order routines grounded in chains of metaphorizings (Sfard 1997). So-called *abstract thinking* could refer to just mechanical symbol manipulation following syntactic rules (requiring neither deep understanding nor visualizing), as in rote algebraic calculations. More likely, though, it could refer to *zooming out*, seeing the "big picture," or putting under a common umbrella various hitherto disconnected phenomena that we re-interpret as different avatars of a same notion, which we construct via a constellation of metaphorizings, among which we move. In any case, we claim that abstract thinking without (overt or covert) metaphorization is blind. It could lead us anywhere among the sundry possible mathematically correct developments.

« 29 » Regarding **Norton & Kokushkin**'s **Q2**, we may point out that we did not *define* metaphors as inference-preserving from one inferential domain to another. We said that metaphors *may be metaphorized* as such mappings. Recall Paul Ricoeur claiming that we cannot refer to metaphor other than metaphorically (Ricoeur 1975). As put forward in our joint work with Varela on self-reference (Soto-Andrade & Varela 1984), this sort of circularity is not to be shunned but to be acknowledged as a fundamental property of living systems. Our experimental epistemological observations suggest the "arrow metaphor" for metaphorization when we observe students looking at a phenomenon and seeing something different from what they acknowledge was presented to them. This kind of mapping seems to be feasible because of a built-in cognitive mechanism of our species, as Stanislas Dehaene and collaborators have shown (Knops et al. 2009). In our view, the nowadays too popular metaphor of (an underlying) "structure" is not necessary to explain the working of metaphorizing. The latter is better described (metaphorized) in terms of processes or dynamical systems (Gibbs 2011), rather than in terms of structures or schemes, which are an *a posteriori* construction.

« 30 » So, we do not agree with **Norton & Kokushkin**'s claim, in §1, that metaphors presuppose abstract structures. We argue that abstraction is grounded on metaphorizing and that abstract structures do not precede metaphorizing but are rather constructed *a posteriori*. Recall our students' metaphorizing of the frog's random walk as a splitting process (§28), which enables them to construct *a posteriori* the abstract notion of probability.

« 31 » Finally, the question arises of providing examples of enactive metaphorizing experiences in the construction or learning of mathematical concepts, as in **Sevim**'s **Q1**, or alternative metaphors for space or number, as in **Norton & Kokushkin**'s **Q3**. Indeed, classical abstract mathematical concepts or theorems are metaphorized and bodily enacted more often than we might think. For instance, the fundamental theorem of calculus can be metaphorized starting from the insight that when one climbs a staircase, the *height* one ascends is just the sum of the raisers of the stairs (which may have varying raisers and treads). Then, if one places a small plank of wood slanted on each stair (so that one could roll one's travel bag upstairs), the raiser of the stair may be recovered from the slope of the plank and the tread of the stair. By shrinking the stair treads to an infinitesimal length, one gets the total height of the staircase as the Riemann integral of the slope function (the derivative of the height function). Analogously, one can metaphorize primitives as areas under a curve.

« 32 » Regarding the concept of number, we remark that its primary metaphors are "number as quantity," and then "number as a location" (on the number line). A less familiar, "Darwinian" metaphor, "numbers as processes," derives from the work of 11th century philosopher Shao Yong, as remarked by Gottfried Wilhem Leibnitz (Ryan 1996). Shao Yong organized, first, the 64 hexagrams constituting Yi Jing, the ancient Chinese oracle, in an 8x8 square (Figure 1), which, from a Western perspective, can be read as the binary coding of the number sequence from 0 to 63.[1] Then, Shao Yong drew the Xiantian diagram (Figure 2),

1 | Cf. our presentation "The binary tree and its avatars: From Xiantian to the eternal symmetree…" at Topic Study Group 27 at the 14th In-

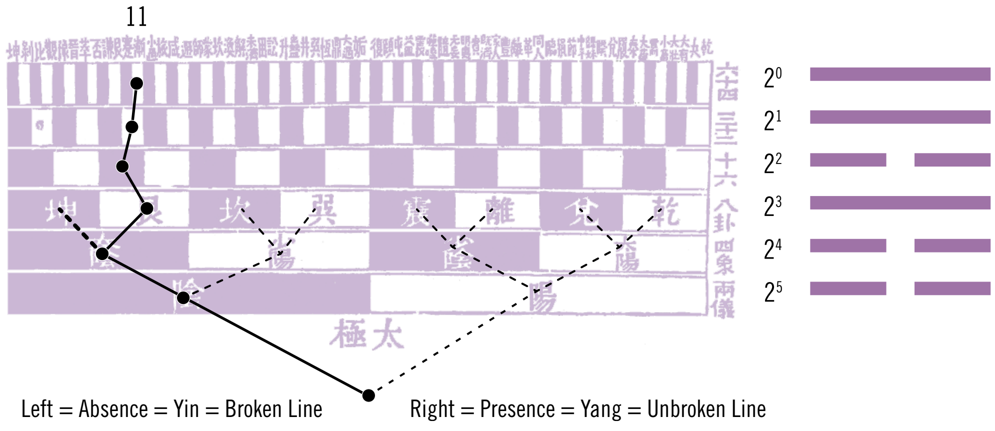


**Figure 2** • Number 11 as an ascending path in Shao Yong's binary tree. (Xiantian diagram modified from https://www.biroco.com/yijing/scans/ztdxiantian1257.jpg)

which presents the hexagrams as the outcome of iterated bifurcations in a six-generation binary tree. Notice that he followed the classical Chinese metaphorical correspondences of Yin = broken line = darkness = black = left, and Yang = unbroken line = light = white = right. This strongly suggests a process generating ascending zigzag walks. In Figure 2, we exemplify this for the number 11 with its associated ascending path resulting from the metaphorical "weighing" of 11 with the help of the binary kit of weights 1, 2, 4, 8, 16, 32.[2]

« 33 » Shao Yong's metaphorical construction of numbers as processes can be extended to all whole numbers if we consider binary trees with an arbitrary number of generations. If we want, however, to metaphorize all whole numbers as ascending paths in a *single* containing infinite binary tree, we need to turn Shao Yong's metaphorizing "on its head," as we explain for the number 11: Since $11 = 1 \times 1 + 1 \times 2 + 0 \times 4 + 1 \times 8$, we see iteratively that $2^0$ is present (i.e., 11 is odd), $2^1$ is present, $2^2$ is absent and $2^3$ is present. So, we now get the ascending path going RRLRLL (where R = right, L = left), which is the "reverse" path of the previous one: LLRLRR (for 3, we get RRLLLL instead of LLLLRR). Now, in the infinite binary tree, all whole numbers appear metaphorically as ascending paths that at some point begin to turn left forever.

« 34 » This metaphorical construction leads us to wonder about the "arithmetic identity" of the other possible ascending paths in the infinite binary tree, i.e., the paths that turn right an infinite number of times, the extreme case being the "far-right" path that turns right forever. In this way, we meet a friendly enactive metaphor for the quite abstract concept of 2-*adic integers* in number theory. They were first constructed in a purely formal algebraic way by Kurt Hensel (Dickson 1910) – motivated by Diophantine equations – as infinite series of non-negative powers of 2. The ascending path associated with such a series is the one that turns right at the $n$-th bifurcation if $2^n$ appears (is present) in the series and turns left if $2^n$ is absent (starting with $n = 0$).

« 35 » Hensel, who calculated adroitly with his power series, would write the far-right path, for instance, as the infinite sum $S = 1 + 2 + 4 + 8 + \ldots$. of *all* powers of 2 (obviously divergent in Calculus 1). He assigned, however, the value −1 to it, because he noticed that when we add an extra 1 to $S$, we see (metaphorically) this extra 1 coupling with the 1 in $S$, affording an extra 2 in $S$, which in turn couples with the 2 in $S$, giving an extra 4 in $S$, which couples with … etc. A "domino effect" occurs, which "kills" all the powers of 2 in $S$, so that $1 + S = 0$, showing that that $S = -1$. This is yet another example of the ubiquitous domino effect metaphor

« 36 » We see that the quite abstract mathematical notion of 2-adic numbers appears to be grounded in our bodily experience of walking along forking paths or engaging in decision trees that unfold in the course of time. Remarkably, here, the binary tree does not appear as just a *static structure,* but as a *generating process.*

« 37 » The examples above strongly suggest that even quite advanced abstract mathematical notions can be approached or even (re)discovered via (chains or constellations of) suitable enactive metaphorizations (Sfard 1997). **Sevim**'s **Q2** may then have a positive answer, in the sense that practically all metaphorizing ultimately arises as a result of bodily motion, supporting the hypothesis that mathematical cognition is fundamentally embodied.

ternational Congress on Mathematical Education (ICME 14), 12–18 July 2021, Shanghai.

2| We call this a "Darwinian metaphor" for numbers, because of its analogy with Darwin's insight that the sundry living species on earth were the outcome of a process suitably metaphorized by the "Tree of Life." Notice here how some cultures facilitate the emergence of some profound metaphorizings for mathematical objects while others rather thwart them.

## References

**Abrahamson D. & Sánchez-García R. (2016)** Learning is moving in new ways: The ecological dynamics of mathematics education. Journal of the Learning Sciences 25(2): 203–239.

**Brousseau G. & Warfield V. (2014)** Didactic situations in mathematics education. In: Lerman S. (ed.) Encyclopedia of mathematics education. Springer, Berlin: 163–170.

**Díaz-Rojas D. & Soto-Andrade J. (2017)** Enactive metaphors in mathematical problem solving. In: Doole T. & Gueudet G. (eds.) Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10), 1–5 February 2017. DCU Institute of Education and ERME, Dublin, Ireland: 3904–3911. ▸ https://cepa.info/6173

**Dickson L. (1910)** Hensel's theory of algebraic numbers. Bulletin of the American Mathematical Society 17(1): 23–36.

**Gibbs R. W. (2011)** Evaluating conceptual metaphor theory. Discourse Processes 48(8): 529–562.

**Knops A., Thirion B., Hubbard E., Michel V. & Dehaene S. (2009)** Recruitment of an area involved in eye movements during mental arithmetic. Science 324(5934): 1583–1585.

**Maturana H. R. & Varela F. J. (1980)** Autopoiesis and cognition: The realization of the living. Reidel, Dordrecht.

**Petitmengin C. (2006)** Describing one's subjective experience in the second person: An interview method for the science of consciousness. Phenomenology and the Cognitive Sciences 5: 229–269. ▸ https://cepa.info/2376

**Ricoeur P. (1975)** La métaphore vive. Éditions du Seuil, Paris. English translation: Ricoeur P. (1977) The rule of metaphor: Multi-disciplinary studies of the creation of meaning in language. Translated by Robert Czerny, Kathleen McLaughlin and John Costello. University of Toronto Press, Toronto.

**Ryan J. A. (1996)** Leibniz' Binary System and Shao Yong's "Yijing". Philosophy East and West 46(1): 59–90.

**Sáenz-Ludlow A. (2004)** Metaphors and numerical diagrams in the arithmetical activity of a fourth-grade class. Journal for Research in Mathematics Education 1(35): 34–56.

**Sfard A. (1997)** Commentary: On metaphorical roots of conceptual growth. In: English L. D. (ed.) Mathematical reasoning: Analogies, metaphors, and images. Lawrence Erlbaum Associates, Mahwah NJ: 339–371.

**Soto-Andrade J. (2018)** Enactive metaphorising in the learning of mathematics. In: Kaiser G., Forgasz H., Graven M., Kuzniak A., Simm E. & Xu B. (eds.) Invited lectures from the 13th International Congress on Mathematical Education. Springer, Cham: 619–638. ▸ https://cepa.info/6219

**Soto-Andrade J., Jaramillo S., Gutiérrez C. & Letelier J. C. (2011)** Ouroboros avatars: A mathematical exploration of self-reference and metabolic closure. In: Lenaerts T., Giacobini M., Bersini H., Bourgine P., Dorigo M. & Doursat R. (eds.) Advances in artificial life: Proceedings of the Eleventh European Conference on the Synthesis and Simulation of Living Systems (ECAL 2011). MIT Press, Cambridge MA: 763–770.

**Soto-Andrade J. & Varela F. J. (1984)** Self-reference and fixed points. Acta Applicandae Mathematicae 2(1): 1–19.

**Varela F. J. (1996)** Neurophenomenology: A methodological remedy to the hard problem. Journal of Consciousness Studies 3(4): 330–349. ▸ https://cepa.info/1893

Received: 12 July 2021
Revised: 19 July 2021
Revised: 20 July 2021