cialists like teacher-observers or researcher-observers, carry an ethical responsibility to consider why they might provide, occasion, and attend to specific ideas. If problematizing, as a conceptualization of student problem solving activity, is to improve its relevance to the mathematics classroom, it must evolve to consider that teachers do not observe student journeys with the sole purpose of describing them; teachers live with student journeys in relationship to anticipated goals, thus creating an ethic of intentionality in their interactivity. Acting with an intentional ethic requires the teacher to appreciate the particular potency, as a specialist, that their provisional, occasional, and attentional moves contain, and mobilize that influence to amplify and accentuate specific activity toward goals while remaining responsive to the needs of the particular situation.

« 10 » Gandell and Maheux provide a useful conceptualization of the interactivity of student knowing as they slowly walk toward a fountain, one whose applicability to the mathematics classroom is inversely related to the ethical distance of the observer. Observing the drifting nature of student mathematical knowing must embrace the important role a teacher-observer plays in shaping student knowing – in walking *with* students – and sharing the "many different pathways to the fountain, many different experiences along the way and many different fountains at which to arrive" (§43) rather than simply meeting them at their destination.

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## Interacting with Other People's Boundaries, Remainders, and Static Enclosures

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**> Abstract** • In this commentary, I point to a few ways that Gandell and Maheux's important work on problem posing|solving could be further extended. I also offer some pedagogical insights that tie the authors' ideas to my own experiences as an educator and researcher.

« 1 » In their target article, Robyn Gandell and Jean-François Maheux integrate Proulx and Maheux's (2017) problem posing|solving construct with Tim Ingold's (2009) journeys, lines, pathways, and knots, in order to explain how a group of students' mathematical problems evolve and unfold over time, and how the students' problematizations entwine with one another. The posing|solving of problems in mathematics is seen as a continuous, enmeshed, enactive, dialectical and wayfaring journey, in which students' interactions with the given prompt, one another, and their environment, are coordinated.

« 2 » In their attempt to understand the mathematical activities of their students from an enactivist perspective, according to which all knowing is doing, Gandell and Maheux appropriately use Ingold's (2009) dynamic views of place, movement, and knowledge. They provide a fine-grained analysis of how a group of students interact with a carefully designed modular arithmetic task that encourages them to explore number patterns and specify their own problems without selecting predetermined (mathematical) strategies. In doing so, Gandell and Maheux extend the work of Proulx and Maheux (2017) by adding insights into how posing|solving, or problematizing, may occur through enactions such as gestures, movements and social interactions.

« 3 » In the given modular arithmetic task, which involves a game of throwing a ball and figuring out various scenarios for different people numbers and different place-jump sizes (§14), all players get to throw the ball when the people number and the place-jump number are relatively prime. In §20 through §41, Gandell and Maheux present an analysis of how the problematization knot of the first throw, or figuring out how the game works, emerges from students' interactions.

« 4 » In Proulx and Maheux (2017), mathematical problem posing is defined as figuring out what seems to be a mathematical problem at any given moment, whereas mathematical problem solving is defined as figuring out how these posings can be seen as mathematical actions. In turns 1–13 of the first data excerpt (§18), although the students are posing problems to enact the game and make sense of how the game works, their problems do not seem to be about them figuring out how their problems (how does the game start, who throws next, …) can be seen as mathematical actions. I agree with Gandell and Maheux that Kit's rhythmic counting has connections to mathematical activities of exemplifying, patterning, and abstracting (§30), but it seems to only offer help in figuring out how the throws work. As the authors explain in §§31–35, Kit's problematizations leave embodied *traces* that help build the group's collective pathway. On the other hand, turns 15–19 in the second data excerpt do seem to open up future directions and horizons with regard to the unique mathematics of the given game: creating and explaining a shortcut, a mathematical generalization that could be used for any size of people number and place jumps. Thus, to me, as a mathematician, the most interesting part of the article is the analysis of the second data excerpt provided in §§42f.

« 5 » While the analysis provided in §§42f is useful to understand how the group started to make sense of the unique mathematics of the game, it is unclear which problematizations in the final portion of the class session acted as places that students came to and went from, and how the students' collective posing|solving activity evolved into the problematization meshwork of their final solution: "If the number of people cannot be evenly divided by the number of place jumps, as long as the number is greater than 1, everyone will get a turn" (§20). At this point, I wonder: How did the students' exploration of an anticlockwise throw (to one person to the right) lead to their discussion of odd numbers in turns 15–19? Q1 For example, did the students' new lines of motion (§§37–41) involving the anticlockwise throws evolve into further mathematical problematizations of the relationship between the people number ($n$) and the place-jump number ($m$), such as: if $m = n - 1$ then the ball is thrown one person to the right? Or, during the later portions of the class session, did these new lines of motion entwine into problematization knots involving explorations of why $n$ or $m$ need to be odd numbers? Such analysis would help us mathematics educators to better see how mathematical problems are transformed.

« 6 » At this juncture, I would like to point to an idea that may help further our understanding of how mathematical problems are transformed. Anna Sfard (1991) proposes that any mathematical idea can be seen both structurally as an object and operationally as a process. She argues that these two ways of working with mathematical ideas are not in opposition to each other, but are complementary. Providing convincing evidence from both the historical development of mathematical ideas, such as imaginary numbers and functions, and from mathematics education research on how students develop these ideas, Sfard points at the common experience that learners first start interacting with a mathematical idea from an operational or process point of view, and later move to using the same idea more structurally, as a single object. And, "whether the issue of applications or of education is concerned, the operational and structural elements cannot be separated from each other" (Sfard 1991: 9). In a sense, one can say that just as problem posing cannot be separated from problem solving, the process and object natures of a mathematical idea cannot be separated from one another. Sfard further explains how the process view of a mathematical idea is reified into the object view of the same idea; and she makes it very clear that such reification is notoriously slow and difficult.

« 7 » Although one may see objectification or reification of mathematical processes as akin to Ingold's (2009) idea of the logic of inversion, in which lived experiences are turned into placeless boundaries, I suggest that it will be more useful if we consider this issue from a different line of thinking.

« 8 » For example, when I teach college algebra to first-year university students, I present to them mathematical tasks such as "Solve for $x$: $(x+5)^2+(x+5)=4$" and observe how they interact with these equations. Often times, most students express their thoughts of adding five to an unknown quantity and squaring it afterwards. Without a ready-made "pill," most get stuck and cannot go further. Unlike my students, when I see this task, I see the problem: Solve: $u^2+u=4$. What enables me to transform the given task into a much more familiar quadratic equation is my past histories of working with mathematical expressions such as $(x+5)$ and developing an understanding of the expression as a single entity. Instead of thinking about the process of adding two things, I simply think of the sum as a whole. Using Ingold's (2009) ideas of pathways and places, I would argue that my interactions with $(x+5)$ as a single object, and not as a process, is a meaningful place that I come to and go from. I am not transported to a pre-determined destination through a vehicle that prevents me from taking the journey of laying mathematical trails; on the contrary, I am in the midst of my mathematical journey of solving the quadratic equation: $u^2+u=4$. I simply choose to treat $(x+5)$ as a single object, because I do not need to travel the path of adding five and $x$. This way of interacting with the given task, which is a result of my successful active (re)creation of prior passages (§9), is more suitable for producing a better solution to the given problem. Every step of my journey is meaningful to me, and I find it very beneficial to treat the mathematical idea of an expression as an object in this very situation.

« 9 » On the other hand, if I were to give this above method of solving the task to my students as a ready-made strategy, as a "$u$-substitution strategy," and ask them to mimic my steps, then my students certainly would be transported to a pre-determined destination (successful solution) through a vehicle (the strategy) that prevents them from laying down their own mathematical paths, lines of motion, or traces. They would be interacting with the results of my (or other mathematicians') prior mathematical activity: my (or other mathematicians') remainders. Then I would say that Ingold's (2009) logic of inversion, of turning students' potential pathways into static enclosures or boundaries, through which they are transported from one discrete dot to another discrete dot, would be taking place.

« 10 » It is not uncommon to observe teachers encouraging their students to recognize prompts/tasks as categories of problems that require the use of certain pre-existing heuristics or strategies. Many students do not get to pursue their own pathways that have the potential to transform the given prompts into their own problems. Many of them do not lay down traces that emerge from the work on their own versions or their own past pathways. They are rushed to final destinations through fast vehicles of a priori strategies. I have witnessed this many times as a mathematics educator both in my and others' classrooms, and I have studied alternative ways of engaging students in meaningful mathematical activity. When students are given opportunities to explore their own ideas and questions, they may encounter situations that are surprising to them and become more persistent in their pursuits toward a solution (Sevim & Cifarelli 2013; Cifarelli & Sevim 2015).

« 11 » We, as educators, certainly want our students to experience mathematics lessons as mathematical meshworks in which they create and progressively disclose their own mathematical horizons, as opposed to meaninglessly manipulating or mimicking their teachers' static remainders, boundaries or strategies. Using Ingold's metaphors is an excellent way of re-thinking our mathematical pedagogies. Vis-à-vis time constraints and political pressures, in our efforts to help our students, we unintentionally transport them to places without letting them take on a mathematical journey of their own. Mostly worried about seeing our students produce mathematical products that are similar to ours, we present to our students mathematical curricula that are merely networks of transports, and not meshworks of wayfaring (Ingold 2009). I suggest that more mathematics educators conduct research studies similar to that of Gandell and Maheux, which help advance our collective understanding of how problem posing and solving co-evolve.

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# Problem Posing and Solving: Wayfarers Making their Way Along Problem Pathways

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**> Abstract** • Gandell and Maheux present an interesting analysis of the problem posing (PP) and problem solving (PS) of a group of students playing a throwing game. The analysis builds on recent studies that explain how PP and PS co-evolve as students interpret and ultimately complete the tasks that we give them. I will make some preliminary comments on the researchers' enactivist approach to studying PP and PS, draw some comparisons with constructivist approaches, and then comment on the analysis of the student work.

« 1 » I enjoyed reading the target article by Robyn Gandell and Jean-François Maheux and thinking about their ideas of how problem posing (PP) and problem solving (PS) co-evolve in the course of action. The article provides a nice follow-up to those studies that have documented connections between problem posing and solving (Cifarelli & Sevim 2015; Proulx & Maheux 2017) as well as studies that call for radical reformulations of the ways researchers consider problem solving and posing (Lesh & Zawojewski 2007; Ärlebäck & Doerr 2015).

« 2 » While the target study extends interest in the research on PP and PS, the researchers have a more ambitious goal than merely to replicate and confirm what others have found. By observing a small group of students solving a non-traditional open-ended mathematical task, the researchers look to not only extend the findings of Proulx and Maheux (2017) to open-ended tasks, but also to offer a fresh explanation of how problem posing and solving can commence as a shared experience within a group of students. For this, they draw from Tim Ingold (2009) to develop a metaphorical characterization of problem solvers as wayfarers for the reformulation of PP and PS. These are very ambitious goals to be sure.