# Problematizing: The Lived Journey of a Group of Students Doing Mathematics

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**> Context** • Mathematical problem solving is considered important in learning and teaching mathematics. In a recent study, Proulx and Maheux presented mathematical problem solving as a continuous dialectical process of small problem posing and solving instances in which the problem is continuously transformed, which they call problematizing. This problematizing conceptualization questions many current assumptions about students' problem solving, for example, the use of heuristics and strategies. **> Problem** • We address two aspects of this conceptualization: (a) how does problematizing evolve over time, and (b) how do the students' problematizations interact? **> Method** • In this study, we apply and further develop Proulx and Maheux's enactivist perspective on problem solving. We answer our questions by applying micro-analysis to the mathematical problematizing of a group of students and, using Ingold's pathways and meshwork as our framework, illustrate the lived practice of a group of students engaged in mathematical problem solving. **> Results** • Our analysis illustrates how mathematical problematizing can be viewed as a complex, enmeshed and wayfaring journey, rather than a step-by-step process: in this enactive journey, smaller problems co-emerge from students' interactions with one another and their environment. **> Implications** • This research moves the focus on students' mathematical problem solving to their actions, rather than strategies or direct links from problems to solutions, and provides a way to investigate, observe and value the lived practice of students' mathematical problem solving. **> Constructivist content** • Our work further strengthens the understanding of mathematical activities from an enactivist perspective where mathematical knowledge emerges from interaction between individual and environment. **> Key words** • **Problem solving, problematizing, journey, heuristics, emergent, enactive, movement, rhythm, traces, artefacts.**





*It was a merchant who sold pills that had been specially designed to quench thirst. […] 'It's a great way to save time,' said the merchant. 'Experts made calculations. With these pills, we save fifty-three minutes in every week, and we don't feel the need to drink anymore' […] The little prince said to himself, 'If I had fifty-three minutes to spend as I liked, I would slowly walk towards a fountain.' (de Saint-Exupéry 1945: 86, our translation)*

## Problem solving to problem posing

« 1 » Mathematical problem solving is important in the learning and teaching of mathematics and appears in the research and curricula of many countries (Bazzini, Sabena & Villa 2009; Santos-Trigo & Gooya 2015; Schoenfeld 1992). Increasingly some assumptions about problem solving, including what constitutes a problem, are being questioned (Schoenfeld 1992; Singer, Ellerton & Cai 2013). Mathematical problem-solving processes used in classrooms are often inspired by George Polya's (1957) four-phase problem-solving model (understand, plan, carry out, look back) and involve various kinds of heuristics and strategies (Duncker 1945; Garofalo & Lester 1985; Schoenfeld 1992; Carlson & Bloom 2005). Although these strategies are used in classroom instruction, research seems to show that they do not always transfer to different types of problems and may not improve students' more general problem-solving skills (Lester & Kehle 2003; Schoenfeld 1992; Mayer & Wittrock 2006). Moreover, classroom implementations of Polya's model often seem to turn the richness and complexity of students' problem solving into a step-wise process (Griffin & Jitendra 2009; Riley, Green & Heller 1983; Marshall 1995) and may, like taking a pill, create shortcuts that save time but at the expense of losing valuable problem-solving experiences, such as the exploration of mathematical ideas. So how may we capture the lived journey of students' mathematical problem solving: and like Antoine de Saint-Exupéry's Little Prince "slowly walk towards a fountain"?

« 2 » Some researchers acquainted with enactivism are also interested in problem solving. For example, this journal has offered a discussion of students' representations and conceptualization in problem solving (Cifarelli & Sevim 2014), which generated a number of equally interesting commentaries connecting it with questions related to ontology and the role of the students' activity in the learning process (François 2014); issues regarding individual and collective learning (Goodson-Espy 2014); and students' refining approaches (Castillo-Garsow 2014). A few years later, Jerome Proulx and Jean-Francois Maheux (2017) reactivated those discussions by drawing attention to students' problem *posing* as an emerging notion in the field (Singer et al 2013; Barabé & Proulx 2015). Their interest in spontaneous questions or mathematical activities that freely arise from a given prompt described as the continuous transformation of prompts into problems, simultaneous problem posing and solving they call "problematizing" (Maheux & Proulx 2014), also generated rich reactions. To recall some, Nat Banting and Elaine Simmt (2017) extended the concept to consider the observation and direction toward more or less specific "goals," Laurinda Brown (2017) added some insights regarding embodiment, emergence, and intersubjectivity, and Matthew Harvey (2017) commented on how their perspective enables us to better account for the organism-environment coupling described in enactivist theories. Our intention here is to offer an enriched conceptualization of mathematical problematizing, integrating many of these distinctions and elaborations. We further develop enactivist conceptualizations drawing on the concepts of emergence and coordination at the heart of Humberto Maturana and Francisco Varela's (1987) seminal work, revisited through distinctions offered by anthropologist Tim Ingold (2009), interpreted in the context of mathematical problem solving.

« 3 » In this contribution, we will address two questions that arise from this previous work: (a) how does problematizing evolve over time, and (b) how do the students' problematizations interact? Specifically drawing on the concepts of emergence and coordination, we suggest adding to Proulx and Maheux's problematizing conceptualization using Ingold's (2009) concepts of pathways and meshwork. We use these concepts as a framework to analyze a group of students problem posing and solving and to illustrate the lived life of their mathematical problematizing as a continuous, enmeshed, enactive, dialectical problem-posing and -solving process.

## The emergent nature of problems

« 4 » From an enactivist perspective, knowledge co-emerges through the interactions of an individual and their environment and these actions are inseparably connected, changing both individual and environment (Proulx & Simmt 2013). Knowledge is not interpreted *though* the actions of an individual; rather these actions are seen *as* thinking, which Maturana and Varela (1998: 26) express as "all knowing is doing and all doing is knowing." Coming to know is a constant process of co-evolving as learner and environment interact and through this process each individual brings forth their own world of significance (Proulx & Simmt 2013), with these worlds "brought forth in co-existence with other people" (Maturana and Varela 1998: 239).

« 5 » Problems, from an enactivist perspective, do not exist as ideas out in the world waiting to be found but *emerge* through the interaction of an individual and their environment (Barabé & Proulx 2015). A prompt in the environment can elicit a response in an individual bringing forth a problem. Although the response is dependent on the individual and their unique history and the prompt and its framing, the response is not in the individual or the prompt but in the interaction between them (Proulx & Simmt 2013): "we specify the problems we encounter through the meanings that we make of the world in which we live" (Proulx and Maheux 2017: 161). A teacher might present a prompt but this only becomes a *problem* for students when the students work on their own version or understanding of it (Simmt 2000). For example, as Proulx (2013) points out, the prompt "solve for $x$: $2x-2=0$" can become a problem in a variety of ways and may lead to a variety of mathematical activities (arithmetical, geometrical, algebraic). The particular activity that emerges is brought forth from both prompt and individual, thus demonstrating the emergent nature of mathematical problems. From an enactivist perspective, prompts are not problems but are triggers, provoking students to act. The outcomes that arise from these prompts, the problems that are posed and solved, depend on the student, the prompt and the environment.

« 6 » For Maheux and Proulx (2014) posing and solving are not distinct events but a continuous process in which every mathematical action transforms the prompt into a new prompt. By transforming $2x-2=0$ into $2x=2$, a different problem emerges, which becomes the new prompt for another transformation and the first of many "mathematical moves" (ibid: 37). From this perspective, problems, solutions and strategies seem to be artificial categories, often devised to explain mathematical actions in problem solving a posteriori. Proulx and Maheux instead propose a continuous dialectical process in which problems are both posed and solved at the same time,

> "[…] conceived as two-sided, constitutive of both, inseparably, a posing and a solving: a posing|solving, where the Sheffer stroke between posing and solving insists on the co-constitutive aspects of each." (Proulx & Maheux 2017: 163)

« 7 » Our interest is in two new aspects of Maheux and Proulx's mathematical problem posing|solving, which they call *problematizing* (Maheux & Proulx 2014): the unfolding of mathematical problem posing|solving over time; and how students' problematizations entwine. As with Creative Problem Solving (Treffinger 1995), we are interested in a more descriptive view of problem solving, "moving away from traditional, linear prescriptive step- or stage-sequential models" (ibid: 301). Closer to the enactive perspective of the emerging nature of problems, our focus, on coordination in problematizing, derives from research, such as that of Turner, Gutierrez and Sutton (2009), which shows that group problem-solving activity, although not necessarily continuous, arises out of all open-ended (multiple-strategy, multiple-solution) problem-solving tasks.

« 8 » In order to investigate these two new aspects, we examine of a group of students engaging with a mathematical prompt. We analyze the students' coordinated posing|solving from an enactivist perspective, in which students continuously pose and solve their own problems and where we observe the students' actions (including verbalizations, gestures, and full-body movements) as forms of knowing, following Maturana & Varela's (1998: 26) aphorism: "all knowing is doing and all doing is knowing." By focusing on the emergence of the students' actions within their environment (Maheux & Proulx 2015), we call attention to the non-linear, enmeshed and emergent nature of these students' problematizations. Existing accounts of divergence in the work of (small groups of) students doing problem solving in classrooms often cast these as independent "learning trajectories," with a static focus on what students know or think at a given moment, assuming hierarchical stages of learning (e.g., Ellis 2014). In the next section, we explain how Ingold's (2009) conceptualization of movement and places will help us analyze such problematization as a dynamic process, with students moving in multiple directions: an approach better suited to enactivist thinking.

## Ingold's movement and places

« 9 » Although Maheux and Proulx's research often uses the language of movement to talk about problematizing, how these movements interact in mathematical activity has yet to be fully explored (Roth & Maheux 2015). One way to do this is by thinking about students' problem solving as a journey: a traveling along a path across a "problem," or perhaps an expedition triggered by the will to find, to know or to discover something. Ingold (2009) describes living life as moving along continuous, enmeshed pathways, to, from and through places, towards progressively disclosed horizons. His analysis draws on his study of *lines* as processes instead of static objects, and the description of how wayfarers' pathways are not like following a trail but the active (re)creation of passages where "only upon reaching his destination, in this case, can the traveler truly be said to have found his way" (Ingold 2007: 16). This conceptualization allows us to observe the movement of problematizing in terms of how it unfolds and entwines over time.

« 10 » Moreover, Ingold's work offers a useful distinction between a movement and its remainder, the trace its leaves behind. For example, he explains how once a hand has drawn a circle on paper, we see only the static enclosure of the completed line: substituting the continuous movement of the hand for a static shape. By the "logic of inversion" (Ingold 2009: 29) we have changed a continuous pathway into a static enclosure: "[we] turn the pathways along which life is lived into boundaries within which it is enclosed" (ibid). Within this "logic of inversion" we are transported from place to place with no discernible journey between these places. The notion of using a *strategy* seems similar to this idea of transportation between places. For Ingold, however, life is lived through the journeys wayfarers take along wandering pathways between places: "lives are led not inside places but through, around, to and from them, from and to places elsewhere" (Ingold 2000: 229). From this perspective, heuristics seem to be a priori or a posteriori recollections of mathematical journeys. Thus, Ingold helps us see how limiting problem solving to heuristics and strategies turns "movement in which action and perception are intimately coupled into [an experience] of enforced immobility and sensory deprivation" (Ingold 2009: 38): the slow walk to the fountain becomes taking a pill.

« 11 » Ingold also contrasts the notion of space with the idea of inhabiting places: a land, a country, the Earth. Lives, for Ingold (2009), unfold not in places but as interwoven pathways forming a meshwork. Where travelers meet, their pathways entwine creating knots in the meshwork. These knots are moments we tend to locate in space when we see them as static and confined: a house, for example, is a space. However, for Ingold, it is not the house that matters, it is the series of activities, the knot of pathways taking place that make the enclosed space of the house become an inhabited place, a home. Knots are defined by the movements through and around, and to and from them and are therefore changing and changeable. So, we are invited to look at problem solving not in terms of "stages" (or "steps" like posing, representing, planning, solving), but to examine how moments of problematizing occur, and how students move through and around, to and from moments of problematizing. Wayfarers, Ingold insists, are not restricted by boundaries of places in space but are moving towards horizons that are also ever changing and unfolding as pathways and knots interweave. Maybe we could similarly analyze students' problematizing as inhabiting a mathematical land.

« 12 » In the following analysis we are interested in the movement of students' lived lives: the pathways that they make, the places they move through, and the knots they create, as they engage in problematizing, not the final destination or the strategy (transport) they select. Using Ingold's (2009) journeys, pathways and knots helps us to investigate the dynamic intricacy and lived problematizing of a group of students as they explore a mathematical prompt. This conceptualization allows us to observe the openness, variation, and wayfaring, and the joining and parting, in the students' mathematical activity. We can also observe the enactive nature of the students problematizing, how problems co-emerge from the coordination of students with one another, their environment and the prompt. As the analysis progresses, we observe the students' problematizing as pathways, how these problematizing pathways enmesh to form knots and a meshwork and how the prompt acts as a situated horizon rather than boundary to their problematizing.

## Prompting mathematical activity

« 13 » In this research, four students are recruited as volunteers from a general tertiary bridging program with ages from 18–22 years. As this research is also part of a study into how students use their *whole body* in mathematical activity, the students are in a large room, without tables and chairs, and are invited to move around during the hour-long session. The students are presented with a prompt printed on A3 paper and posted on a magnetic whiteboard with whiteboard pens and magnetic counters: an investigation of the patterns created when a ball is thrown around a group of people (Box 1).

« 14 » In the following brief mathematical analysis of the prompt, as a modular arithmetic problem, for any given game the *people number* is the modulus $n$, counts (*places*) are 1 to $n$, and the *place-jump number* is a repeated addition (multiples $m$). For example, in the three-*place-jump* game, with four *people*, the multiples of 3 (3, 6, 9, 12, …) are congruent to all the possible *places*, 3, 2, 1, and 4 (modulus 4), in the game so that every person will throw the ball. In a two-*place-jump* game with four *people*, the multiples of 2 (2, 4, 6, 8, …) are all congruent to either 2 or 4 (modulus 4). As *places* 1 and 3 are not congruent to multiples of 2, not everyone will throw the ball in this game.

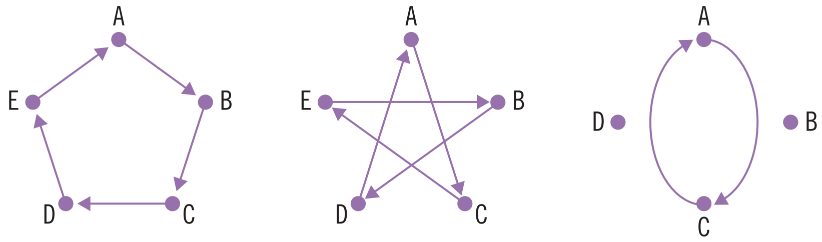
« 15 » In any game, if $m$ and $n$ share a common factor, the multiples of $m$ will be congruent to only some (not all) of the numbers 1 to $n$ and so not everyone will throw the ball. For example, six *place jumps* with nine *people*: the common factor is 3 and all the multiples of 6 (6, 12, 18, 24, …) are congruent to 3, 6 or 9 (modulus 9) so not everyone will throw the ball. A general solution could be: As long as the people number and the place-jump number do not have a common factor, everyone will get the ball (or move in the dance).

« 16 » The following analysis looks at how a group of students mathematically problematize together. We show a way of conceptualizing the students' unfolding problematization, by analyzing what students do in the way Ingold (2009) suggests, and illustrate the possibility of using Proulx and Maheux's (2017) posing|solving conceptualization over time. To capture the full sense of students' problematizing we discuss the students' activities through their enaction of their problematizing, which includes use of speech, diagrams, gestures and whole-body movements. By doing so, we make room for the lived lives of students doing mathematics, which includes, at times in this fragment, acting out a game from the prompt. Although some research pays special attention to how body movements are important in mathematical activity, for example, Luis Radford (2003), this is not our focus here. We choose to use this fragment mostly because the students' body movements help us see their mathematical work more clearly than other moments when the students only write numbers or draw diagrams.

### Box 1: Prompt for investigation

Nic watches a game where a ball is being thrown around a group of people in a clockwise direction. The number of people in the group is called the *people number*.

Each time the ball is thrown in a game it is thrown in equal size *place jumps*. Each person throws the ball to the person on their left the same number of *place jumps* away. When the ball gets back to the first person the game ends.




5 people 1 place jump | 5 people 2 place jump | 4 people 2 place jump

In some games (like the 5-people 1-place jump game and the 5-people 2-place jump game) Nic notices that all the people throw the ball. In other games (like the 4-people 2-place jump game) only some people throw the ball. Nic wonders whether everyone gets to throw the ball in a 4-people 3-place jump game and a 6-people 3-place jump game.

Nic wants to make a dance using this game with people moving between each of the positions instead of the ball being thrown. Nic wants to know if everyone gets to move for different size *people number* and *place jumps*.

Create and explain a shortcut that Nic could use for any size of *people number* and *place jump* size.

Present this shortcut in the last 5 minutes of the session.

« 17 » The students begin the forty-minute session standing in front of the whiteboard, reading and discussing the prompt using the prompt diagrams and magnetic counters. After one minute, Paige is the first to verbalize, "what's place jumps mean?" Kit models the four-person three-place-jump game, then a six-person, two-place-jump game, using counters on the whiteboard, however, his answers to the same game are not consistent. The other group members focus on the board adding utterances and moving counters.

« 18 » Six minutes from the beginning of the session, Chas utters "can we practice the first one … to work it out?" while stepping back into the room behind him. All the students then move into the open room to enact the *four-person three-place-jump* game while Ala offers and brings a counter to use as the "ball." Figure 1 shows how they arrange themselves in the room. In the following fragment we see the four students enact and discuss the *four-person three-place-jump* game.

1. Ala: [*gestures as if to throw the counter*] Can you catch, Paige?
2. Chas: But it's a three-person jump, right?
3. Ala: Yeah.

4a. Paige: So what?

4b. Chas: So you'll [*points to Ala*].

4c. Kit: So [*pointing to Ala with his right extended arm*]

5. Kit: One [*using his right arm, points to himself*], two [*points to Paige, extending arm*], three [*points to Chas with fully extended arm*]. So you'll pass to Chas [*bends his arm into himself then extends it to point to Chas*] and Chas will go, so just pass to Chas. So just pass one person to your right… [*Kit gestures with right wrist to the right then Ala throws counter to Chas*].
6. Paige: [*looks back and takes one step towards the whiteboard*]
7. Kit: And one, two, three, would be Paige [*points to Paige*], Paige would throw to me [*points to himself*], I'd throw to Ala [*points to Ala*].

8a. Ala: [*points anticlockwise around the group from Paige back to herself*]

8b. Chas [*points around clockwise from Ala to Kit then Paige with his right arm and then verbalizes*] Basically it's just going backwards [*gestures counterclockwise with his left hand*]

9. Paige: [*steps closer to the whiteboard*] Doesn't it say left though?

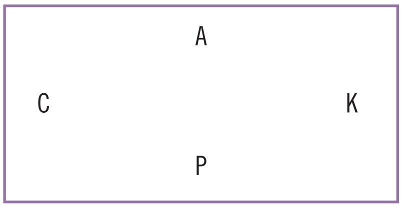



**Figure 1** • Paige, Chas, Ala and Kit distribute themselves in the room. (Faces have been blurred to protect the participants' anonymity).

**Figure 2** • The students working on the whiteboard.

▶▶ 10. Chas: [*turns his body, steps and moves his gaze towards the whiteboard*]

11a. Kit: So you throw it in a clockwise [*turns his upper body slightly left, gestures in a large vertical clockwise circle with his right arm, finishing by leaning right towards his arm]* but three actually equals an anticlockwise throw [*using a horizontal circular gesture with his left arm*].

11b. Ala [*steps towards Paige and Chas, closer to the whiteboard, then steps back as Kit finishes speaking*].

12. Paige: Okay [*turns back away from the whiteboard. Chas and Paige move back to the circle and the counter is thrown from Chas to Paige to Kit and back to Ala*].

13. Ala: [*As the counter is thrown, she points upwards with her right index finger*] Hang on, I thought it was three people.

« 19 » Following this fragment, the students return to stand around the whiteboard. At nine minutes, thirty seconds, they move into the open room to "try dancing" both the four-person, three-place-jump game and the six-person, two-place-jump game then return to the whiteboard where they model games using counters and write *solutions* (Figure 2) on the whiteboard. At twenty minutes, the students are standing around the whiteboard as the following fragment occurs:

14. Paige: Just need an odd number
15. Ala: Yeah of people ...
16. Paige: Of people or place jumps
17. Chas: Three two works [*uses the counters to play three-two game*]
18. Paige: So it always has to be ...
19. Kit: Yeah that works.

« 20 » The students continue for a further twenty minutes, writing on the whiteboard, focusing on creating a shortcut that could be used to determine whether everyone gets involved for any number of people and place-jump size. The group finally settle for the following statement: "If the number of people cannot be evenly divided by the number of place jumps, as long as the number is greater than 1, everyone will get a turn."

## Problematizing pathways and meshwork

« 21 » Our first experience of the students problematizing pathways in the first fragment occurs in a very physical way, as the students move from the whiteboard to arrange themselves in the room (Figure 1). Here we see the students transform the prompt into the problematization of how to enact the game. By continuously, and non-verbally, problem posing|solving their movements, they come to stand with even spacing in a "circle." Ala seems to continue this problematization pathway in the first utterance and actions in the fragment when she offers the first *throw* to Paige (turn 1). As Ala holds the counter, the game starts from her so she appears to be problematizing both that the students embody the game and a way to enact this game.

« 22 » Like Ingold's (2009) places, problematizations are not simply a destination, the result of (mathematical) activity, they are also somewhere to go from. Ala's actions in turn 1 pathway away from the problematization of placement in the room to the problematization of first *throw* (although a *throw* to Paige is not *three place jumps*). In turn 2 Chas coordinates with Ala's problematization to problematize the number of *place jumps* in the game. So, we see the students moving along pathways interacting and responding to one another's problematizations, coordinating by moving to and from emerging problematizations.

« 23 » Although problematizations can be places to come and go from, these places are not distinct like dots on a page but part of a continuous problem posing|solving pathway. Chas's utterance in turn 2 both asks a question and checks his assumption is correct, "it's a three-person jump, right?": posing and solving in one problematization. Coordination is again evident in turn 4, with Paige, Chas and Kit's simultaneous utterances and gestures, which seem to both accept Chas's problematization of *place jumps* and offer the next problematization, indicating where the first *throw* goes. Kit, in turn 5, continues the emerging problematization by gesturing and pointing to each *place-jump* and *throw* position, finishing by simultaneously posing and solving "just pass one person to your right." As the fragment progresses, problematizations unfold not as distinct places, nor as separate posing and solving, but as a continuous, shared, coordinated wayfaring journey.

« 24 » Gradually, as the students intertwine and coordinate their actions and responses with one another, their pathways form a meshwork of problematizations. This can be seen in a number of ways: in turn 2, when Chas both accepts Ala's problematization and challenges her suggestion; in the use of horizontal gestures by Chas and Ala (turn 8), which mimic and enmesh with Kit's actions in turn 7, problematizing *place jumps* and *throws*; and in turns 9 to 11, where Page, Chas and Ala's body orientations and gazes enmesh the problematization of counter direction as they physically move away from enacting the game, turning to look, then move towards the whiteboard. By looking at problematizing in terms of an emerging meshwork, problematizing in a group or over time seems to be a form of coming and going together as the students respond to, mimic, challenge and suggest ideas to one another.

« 25 » The students' unique histories also emerge and entwine through their coordination within this meshwork of problematizations. These histories might include physical experiences, such as playing ball games and using a whiteboard, and mathematical histories such as encountering similar prompts, diagrams, and mathematical ideas (counting, patterns, backward or clockwise movements). The problematization pathway we observe here may entwine with the students' previous meshwork, of lines coming from and going to places elsewhere, including pathways *from* their own previous experiences, such that these problematizations could be very different with different people.

« 26 » The students also appear to move towards another larger question, a statement from the prompt, looking for *a shortcut Nic could use*. Ingold's places are not bounded, but allow for growth and movement "with no walls, only the horizon progressively disclosed to the traveler" (Ingold 2009: 31). The prompt, the first *place* the students inhabit, then provides an opportunity for movement, with a direction that is only "progressively disclosed" (ibid) as the students engage with the prompt and one another. Emergence here is not only in the sudden, ecstatic appearance of something noticeable in the environment (Maheux & Roth 2011), but also in the continuing being and becoming of something turning into something new.

« 27 » Seeing this meshwork of wayfaring in the way Ingold suggests, these students' coordinated actions are not only just the expression of a thought, the completion of agency pairs or the implementation of speech plans. They are the communalizing, the coming from and going to Ingold's places, creating pathways that entwine into a meshwork of problematization. From an enactivist perspective, this emerging meshwork demonstrates the co-evolving of knowing/being/doing, as individual and environment interact. Problematizing in a group and over time is the active work of responding to one's own and others' problematizations and to the environment: moving through, around, to and from places like the Little Prince's walk to the fountain. To fully appreciate the mathematical quality and co-evolution of these problematizations, we need to go into greater detail, and examine what given lines of wayfaring bring, and bring forth.

## Problematization knots and traces

« 28 » As Ingold's (2009) wayfarers proceed along pathways, their lines of movement meet and entwine, forming *knots*. It is these lines of wayfaring and knots that form Ingold's meshwork: "the more that life-lines are entwined, the greater the density of the knot" (Ingold 2009: 33). We see problematization knots forming in turns 2 to 4, as Chas restates that the game requires three *place jumps*. With her utterance "so what …" (turn 4a) Paige seems to both accept Chas's problematization and question what that means for their enactment of the game. At the same time, Chas and Kit point to Ala, both uttering "So…" (turn 4b and c), which seems to simultaneously pose the "problem" of *how does the game start*, and solve this problem by indicating *it starts from Ala*. These actions demonstrate Proulx and Maheux's (2017) conceptualization of problematizing: posing and solving all at once and transforming the *problem*, from "how to enact the game" to "where does the counter go next." So here we see Kit, Chas and Paige's problematization paths entwine as they define and question the first moves in the game to form a problematization knot around how to enact the game.

« 29 » In turn 5, Kit further entwines this knot by coordinating with Ala and the environment, using both movement and verbalisation, to define the first *throw*. For Ingold what we know are not fixed and static things: "A fact simply exists. But for inhabitants, things do not so much exist as *occur*" (2009: 41). Kit's actions in turn 5 suggest he is attempting to make three *place jumps* and the first counter *throw* occur. Kit counts and points three place-jump positions from Ala then utters "so you'll pass to Chas" (turn 5). He then repeats "so just pass to Chas. So just pass one person to your right" (turn 5).

With this repetition and following pause, Kit seems to insist that Ala carry out the action of throwing the counter to Chas, emphasizing the importance of carrying out the physical action of passing the counter, thereby making the throw occur. Making the throw occur can be seen to increase the density of lines in the problematization knot, entwining verbal, gesture and movement pathways. We also become aware that emergence is not about sudden appearance but can also concern persistence.

« 30 » Furthermore, focusing on Kit's speech and gesture in turn 5, we notice a regular rhythm in Kit's counting and pointing that seems to both bring forth and emphasize the regular aspect of three place jumps. For Wolff-Michael Roth and Alfredo Bautista (2012), rhythmic counting foregrounds, organizes, and helps understanding of mathematical patterns by establishing the expectation of future occurrences. This predicting of how a pattern will continue after actions stop is an important part of the process leading to algebraic generalization (Radford, Bardini & Sabena 2006). Thus, Kit's rhythmic counting connects to activities (exemplifying, counting, patterning, abstracting) we recognize as mathematical work, further increasing the density of the problematization knot by entwining and corresponding with pathways that come from and pass through previous mathematical experiences. Rhythmic counting can offer directions for future patterns and mathematical work, and provides another way for the pattern to *occur*, emerging from the coordination of multiple elements.

« 31 » Kit's actions in turn 5 can also be considered as a way of leaving *traces*: the artefacts of the mathematics thinking we do as we think mathematically (Roth & Maheux 2015). Traces, which can include speech, movements and drawings, do not always appear as fully developed thoughts and their meaning often arises after they have been produced. This emergent nature of traces reflects the emergent nature of Ingold's (2009) lived journey "inhabitants know as they go" (Ingold 2009: 41). We see traces left throughout this fragment in a variety of pointing gestures in turns 4, 5, and 7; Kit's insistence of the first throw in turn 5; and throwing and catching the counter in turn 12. As they emerge, these traces entwine and coordinate in problematization knots: Kit's place-jump pointing in turn 5 emerges as circle movements in turn 11a, entwining into the problematisation knot of clockwise and counterclockwise *throws*; and naming the trace of Ala's *throw* in turn 5, "just pass one person to your right," changes this trace into a proposition for all *throws* for this game.

« 32 » We pause to clarify here how we relate what students do and the mathematical "thinking" taking place. Adopting a pragmatic approach, what concerns us is not what students might be thinking *beside* their actions, but the *visible* thinking *in the form* of doing: speech gestures and movements that they make available to one another (Roth & Maheux 2015) and that they entwine into a moving problematisation meshwork. The distinction is important because we do not claim here that, for example, Kit's gestures represent his current thinking on the outside, or that they are *necessary* for his own comprehension of the situation: he might very well clearly understand the situation already and simply want to convince others, or maybe he is trying to affirm himself as the group leader, or perhaps it is only a form of epistemic actions (actions without specific intention except for transforming a situation to allow information to emerge, for example, when we randomly move tangram pieces in the hope that "something" will come out of it). "From the perspective of an observer there is always ambiguity in communicative interaction" (Maturana & Varela 1998: 196). Knowing the *meaning* of any given act is not only impossible for us as external observers, but also for the performer of the act because intentions are always commentaries, and action may carry out more than what seems to be intended.

« 33 » On the other hand, Kit's speech and gesture *as* a form of thinking is what is made available to the group (including himself, as an observer of his own actions): it provides something for the group to coordinate with. Following Ingold's metaphor, we are invited not to think about how students may affect one another's private solving process, but how students create wayfaring pathways that correspond and attend to the problematizations they make available to one another as they journey along together. Wondering how Kit's reasoning affects Ala's or Paige's implies the reification of the processes into fixed objects. As a lived, emerging, unfolding process, problematizing requires a different approach: not the interaction of individual problematizations or their conflation to that of the group (e.g., taking Kit's proposition as something everybody in the group agrees with), but the observation of the students' correspondence and attention and the moving problematization meshwork that is formed.

« 34 » Traces are not just an artefact of thinking. For Ingold (2009), things we know are understood not only by our memories but by the intertwining knots of the stories we tell. Kit's emphasis on, and waiting for, the physical action of throwing and catching the counter, provides a trace that is a memory and a story of a three-place-jump action. The physical experience of this throw provides the group with a memory as an embodied trace to interweave into the problematization knot of the first *throw*. As it is embodied and not permanent this trace may seem to be less permanent and more individually variable than a visual artefact, however, in turns 8 and 11, the trace reappears as circular gestures clearly entwining as lines in the problematization knot.

« 35 » Creating the throw as a trace in turn 5 adds to and thickens the knot of problematizations of place jumps and throws transforming these problematizations. The trace reconfigures the *problem* of *where is the first throw?* to *where is the next throw?* providing a new *problem* that is simultaneously solved with Kit's "one person to your right" (turn 5). This transforming and simultaneous posing|solving suggests traces are also problematizations, places that wayfarers come to and go from, which uncovers a directionality and purpose to leaving a trace not explicitly present in traces as unfinished thoughts (Roth & Maheux 2015). Although traces do not, and perhaps cannot, entirely capture mathematical knowing/being/doing, they may nevertheless offer opportunities for the actions that created them to re-emerge and to coordinate with others' actions. We see now the problematization knot of how to enact the game is "a tangled mesh of interwoven and complexly knotted strands" (Ingold 2009: 37). The knot entwines emerging and coordinating problematizations, which we see starting from Ala in turn 1, include Paige, Chas and Kit's actions in turn 4, Kit's utterances, gestures and rhythm in turn 5, and Ala's throw to Chas. As this knot thickens, the problematizations become more familiar and more useful to the group: co-ordination of actions "takes place in a world we share because we have specified it together through our actions" (Maturana & Varela 1998: 239). Kit's final utterance in turn 5, the more general, and simple, idea of passing to the right, evolves from the interactions we see between the students and their environment demonstrating the enactive nature of problematizations where knowledge is not an object to be acquired but to be enacted.

Figure 3 • Students gesture in turns 7 and 8.

## Lines of motion

« 36 » Finding something new, Ingold suggests, requires "looking for signs of another line of motion that would lead you to your objective" (Ingold 2009: 34). This "looking for signs" is not a mere reflexive activity but occurs in the interplay between individual and environment. As the students respond to one another, they transform their problematizations, changing their pathways, journeys and horizons by finding other lines of motion.

« 37 » In turn 7, Kit brings forth two quite different patterns, which demonstrate an important shift in his line of motion. At first Kit appears to recheck and retrace his path to his "one person to your right" (turn 5) generalization by verbally counting one set of place jumps then pointing to the next throw position. From this check Kit then uses "the next person to the right" generalization to point at and verbalize each of the next throw positions, thereby changing the problematization from *how place jumps* work to *where does the next throw go*. Changing his verbal and movement pattern from indicating *place jumps* to indicating *throws* illustrates important variations in mathematical concepts: the movement changes direction from clockwise to anticlockwise; and the counts change from three to one. Mathematically these actions are seen as equivalent but different representations of the same pattern, a key feature of algebraic thinking (Radford et al. 2006), but coordinating such representations is not necessarily easy. Each representation offers the students different ways of problematizing, and different directions towards a progressively disclosed horizon.

« 38 » Although Kit seems to be carrying along the group's investigation path, "laying out a path in solving" (Proulx & Maheux 2017: 160), this path does not appear as a straight line. In turn 8b, Chas and Ala use and transform Kit's pointing movements from turn 7: performing similar movements but with small, significant variations (Figure 3). Where Kit points anticlockwise to each *throw* position using his extended right arm, Ala, using her right hand and wrist, creates a much smaller movement. Chas produces two different movements: first pointing clockwise with his extended right arm pausing at each *place-jump* position; then using his extended left arm to create a horizontal anticlockwise circle. With his second movement, Chas transforms the *throw* generalization brought forth by Kit, in turn 5, into a continuous anticlockwise movement that he verbalizes as "it's just going backwards" (turn 8b). Kit's actions, in turn 7, create new traces and another line of motion. By copying and modifying these traces in turn 8, Chas and Ala appear to both accept the emerging problematization of *throw* positions, instead of place-jump counts, but also bring forth new lines of motion.

« 39 » Examining Chas's actions in turn 8b in more detail, we gain a better sense of the different line of motion he brings forth. By changing from pointing positions with his right arm to a continuous movement with his left arm, Chas offers two separate and distinct traces for the *place-jump* and counter-*throw* patterns, problematizing the difference between the clockwise *place jumps* and the apparently anticlockwise ("backward") counter *throws* for this game. Chas's actions and verbalization change the simplified pattern "pass to the right" (turn 5), to a more general, anticlockwise, direction for throws in the future. This new line demonstrates an opening to a new horizon: a pattern for how the counter can move throughout the game.

« 40 » For Luis Radford, students' actions within a context may be transformed into abstract objects, which he calls a *contextual generalization*: "the abstract objects are hence abstract while bearing contextual and situated features that reveal their very genetic origin" (Radford 2003: 54). By changing traces that were indicators of positions, place jumps and throws into traces that contain no reference to discrete positions and actions, Chas appears to contextually generalize Kit's traces from turns 5 and 7, changing the discrete pointing into a continuous circular movement. Kit then uses these continuous circle traces in an explanation of the difference between *place jumps* and *throws*. Describing the counter movements, he utters "So you throw it in a clockwise," simultaneously performing a clockwise, continuous, vertical circle with his right arm, "but three actually equals an anticlockwise throw" (turn 11a), performing a horizontal, anticlockwise, continuous circle with his left arm. Chas and Kit's continuous circles (turns 8b and 11a) reference the pointing actions used in turns 5 and 7, transforming the students' actions of pointing to place-jump and throw positions into objects, circle traces, that describe a more general pattern for place jumps and throws. For Radford, contextual generalization occurs when action is transformed into what he calls *abstract linguistic objects*. Here Chas and Kit seem to contextually generalize their actions into *abstract movement objects* that both reference those original actions and are used as objects to describe the game. With this abstraction, Kit and Chas are seen to travel in a new line of motion.

« 41 » From their actions, the students appear to inhabit the place of *solving this problem*, their problematization paths entwine and coordinate to create knots, places of problematization from which new lines of motion emerge. In particular, the students coordinate to form knots around the problematizations of how to enact the game, what place jumps are and how they are used in a game, the difference between place-jump direction and throw direction, and creating abstract movement objects. Although the students attend to one another and coordinate their lines of motion they may (re)visit different places. Ala demonstrates this in her final utterance of the first fragment, "Hang on, I thought it was three people" (turn 13).

« 42 » In the second fragment, lines 14 to 19, the students verbalize odd and even number patterns using what appears to be corresponding meanings for place jumps and how to play a game, in this fragment, a three-person two-place-jump game. With their movement around and through the knots in fragment 1, the students brought forth this knowledge of place jumps and playing a game, demonstrating how knowledge "grows in a field of practices constituted by the movements of practitioners" (Ingold 2009: 42). However, this is not fixed knowledge, but knowledge that emerges: the lines from the entwined knots in the first fragment carry along and intertwine with other knots the students bring forth as they investigate odd and even numbers in the second fragment.

« 43 » Paige problematizes that an odd number might be a general pattern in the games. Although Ala accepts this problematization, by uttering "of people" she seems only to accept that the number of people needs to be an odd number. Paige then problematizes that either people or place jumps could be an odd number. Chas picks up this problematization, modeling a three-person two-place-jump game by moving counters on the whiteboard. With his use of counters and his utterance in line 19, Kit also appears to accept the odd-number problematization. We see here that the lines students entwined in fragment 1 are not contained within the problematization knots, but have carried along, and as the students coordinate their actions, they entwine these lines into a new problematization knot around odd numbers. Knots are tied by lines of wayfaring but these lines are not held in a knot but "trail beyond it, only to be caught up with other lines in other places, as are threads in other knots" (Ingold 2009: 33). Using a strategy, taking a pill, may take you to a fountain (solution), and quench your thirst, but there are many different pathways to the fountain, many different experiences along the way and many different fountains at which to arrive.

## Conclusion

« 44 » In the fragments above we see how students problematize by using their discourse and action to bring forth problems, which they pose and solve at the same time (Proulx & Maheux 2017). Although we see the students transform the given prompt into smaller "problems" of their own, this does not create a path straight towards a solution; instead the student's problematizing seems to be a meandering journey towards an unfolding horizon. Ingold's (2009) concepts of pathways and journeys help us investigate this lived practice and enactive nature of students' problem solving. From a mathematical perspective, the students' *answer* is only partial, and may be considered to demonstrate a limited understanding of the problem and its solution. However, we see this as the final problematization of the prompt. After forty minutes of mathematical activity, the students are ready to move on to something else. Perhaps they could go further, or maybe they are not completely satisfied with their answer but it is "good enough" (Zack & Reid 2003) for the time being, for they have other places to go and things to do, other pathways to take.

« 45 » Ingold's wayfaring pathways help us describe problematizing as a journey of emergent activity brought forth as students coordinate and interact with each other and their environment: creating "a world brought forth in coexistence with other people" (Maturana & Varela 1987: 239). In this journey, problematizations co-emerge as students share, change, borrow, join and part with mathematical ideas, bringing forth problematization lines that entwine, forming problematization knots. The students inhabit these places, moving to, from

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and through, and creating what Ingold calls meshwork: problematizing that evolves over time, moving towards a "progressively disclosed" horizon (Ingold 2009: 31). Such an account differs from existing analysis in terms of "trajectories" not only in terms of their epistemological underpinnings: thinking about problematizing as lines of motion suggests a different way to engage with students. Rather than attending to where students are (what they know, or think), knots and lines interest us in the kind of work that is taking place, and the stories of responding to a prompt: stories to be told conserving their situatedness to the fullest, as ways of inhabiting a mathematical landscape.

« 46 » Problematizing is a lived enactive practice, part of mathematical activity that Ingold's ideas help us both observe and analyze in terms of coordination and coordination of coordination. We are aware of the significant limitations of this study attending here to only four students and small fragments of a single session. However, it shows the possibility that our questions, how problematizing evolves over time and how students' problematizations interact, can be addressed in terms of complex, enmeshed and wayfaring journeys. Our observations of the lived practice of mathematical problematizing shows us the importance of this lived journey. Ingold provides us with a set of powerful metaphors to think about students' mathematical activity and shift attention from end products to the activity itself, to doing mathematics. As teachers and researchers of mathematical problem solving, we wonder how these metaphors can affect our practices. Although we cannot provide an answer here, we feel that using Ingold's meshwork as a framework helps us better understand what students do when they mathematically problematize as they "slowly walk towards a fountain" (de Saint-Exupéry 1945: 86).

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