« 11 » Kauffman's proposal thus refers to the white spot of a second procedure, that of gaining expertise. Although this procedure is left implicit, it appears to motivate claims like: "all actors are affecting one another and the whole" (§31). Such mutual contributions differ from the "stable knowledge" that "is equivalent to the production of eigenforms" (§45). This difference confirms the value of local resources. They are touched upon when it is said that "[t]he world itself is affected by the actions of its participants at all levels" (§45).

« 12 » Kauffman refers to cybernetics as "defined in terms of itself." It focuses on the observer observing herself as an eigenform of the negation operator. This suggests that he recognises the white spot: the focus on observation implies that the focus on action is neglected so further work is necessary. The same type of neglect is exemplified by the focus on "theory" in the area of evolution (Stadler 2016). Kauffman's proposal suggests that in both areas one should emphasise the process of gaining expertise – as part of a "struggle." Organisms have experimented for a long time to (collectively) gain expertise in recognising and using local resources to deal effectively with their environment. Knowledge as a resource is a late arrival.

Gerard de Zeeuw is professor emeritus of the University of Amsterdam and Honorary Guest Professor at the University of Lincoln. His professorial appointments started in 1973 with the topic research and research methodology. The appointments changed with applications ranging from andragology, agriculture and business management to architecture and mathematics. He cofounded the IFSR and the Systeemgroep Nederland and is a member of the American Society for Cybernetics and the UK Operational Research Society – and other organisations.

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Author's Response

Eigenform, Action, the Continuous and the Discrete

Louis H. Kauffman

> Upshot • I discuss the epistemology of action and its relation to the continuous and the discrete, relative to eigenform, eigenbehavior and the deeper structure of eigenforms in relation to the construction of our personal and apparently objective worlds.

« 1 » I will begin with a succinct summary of my point of view about eigenform and then explain how this point of view is related to the many questions and statements given by the authors of the commentaries. In his §6, **Gerard de Zeeuw** remarks that the initial act of transformation is very important for the understanding of eigenform. I agree and will discuss the point now in general, and later quite specifically. An eigenform is a fixed point of a transformation. A transformation is an action that occurs in some domain. Thus, an eigenform begins with an action, and an eigenform is invariant under that action. The notions of both action and invariance occur for some observer who may be part of or identical with the domain that is observed. What is invariant, the eigenform, is not necessarily part of the original domain. The eigenform may be a new entity that has arisen for the observer in the course of interaction with the action or transformation that originated the process leading to the eigenform.

« 2 » Without action, transformation, there is no possibility of an eigenform. It is in the presence of change that the observer can understand, indicate or find constancy. Without some form of constancy, there is no way to observe action. Action is seen against some reference that is regarded as constant.

« 3 » We have indicated the epistemology of eigenform by using a symbol T for a transformation. In this symbolism, an entity X is transformed into a new entity $T(X)$. This gives rise to the possibility of a recursive application of T upon an initial state X and the possibility of an excursion into mathematical formalism. The simplest version of this formalism is the writing of the composite symbol $E = T(T(T(\ldots)))$ that can be taken to mean "an indefinite application of T upon an unknown state" or "the action of T upon itself." In the latter interpretation I can interpret that action of T upon itself as being unchanged by the action of T, just as my thoughts of myself do not change the essential notion of myself (and in fact may support it). In this form we have it that E is a fixed point for the action of T, and so E can be seen as an eigenform for T. In this way any action can give rise (by acting upon itself) to an eigenform. Many cognitive eigenforms have exactly this pattern and it is the case that many objects in the worlds of our perception can, in the last analysis, be identified with actions that are associated with them.

« 4 » There is always the possibility of examining a recursion that is associated with an action. For example, the North Atlantic Ocean acts upon the coastline of Britain over many years so that the coastline becomes an approximate (it is still changing) eigenform for this action. Once approximation appears we enter into mathematical considerations where matters of limits can be quite subtle and the many conceptual domains of mathematical thought need to be examined once again. One of the largest themes related to eigenform then becomes the relations of the continuous and the discrete. These themes are touched upon in my essay with the description of the binomial distribution and its limiting form, the continuous normal curve.

« 5 » **Jean Paul van Bendegem** takes me to task for dealing too loosely with the limiting properties of the binomial distribution. I plead guilty to this accusation! I did not intend to write a mathematical article. My intent was to point out that is well-known to those who study probability and statistics: that the *form* of the normal distribution is approached by the *form* of the binomial distribution. I was not asserting any facts about the point-wise limit of the one and the other. The forms of these distributions can be seen from their graphs and from experiments with balls dropping through lattices. But to actually analyze the limits of the binomial distribution in relation to the normal distribution is a technical mathematical task that was not my intent in the target article. I am tempted to include here some of the details that I like about the mathematics of these limits but I shall forgo the pleasure. The point I want to make is that the two distributions, one continuous and the other discrete, are related not just by form but also quantitatively in that for a toss of a large number of coins we can make quite accurate predictions about their behavior by using the continuous normal distribution. How this can be done is the subject of basic statistics and is a hard-won mathematical subject.

« 6 » **Van Bendegem's** objection points out that if we are to make full use of the concept of eigenforms we shall have to respect the mathematical structures that lie behind them. This does not only apply to the binomial distribution but even to the most general eigenform that I have indicated in §3 above. To make mathematical sense of $E = T(T(T(\ldots)))$, we need to work in a number of directions, as I have in previous papers, using notions of infinity or using notions from the lambda calculus of Church and Curry (Barendregt 1984). **Van Bendegem** objects to the idea that there might be domains where there are always eigenforms. One could agree in concrete mathematical contexts. For example, the Brouwer fixed point theorem: A continuous mapping of a closed disk to itself has a fixed point. If we did not take the closure hypothesis for the disk, this theorem would be false, and it would be inaccurate and uninformative to claim otherwise. But in a reflexive domain D where D is in 1-1 correspondence with the actions of the elements of D, then every element of D has a fixed point by the beautiful construction of Church and Curry: Let A be an element of D. Define $Gx = A(xx)$. Then $GG = A(GG)$. This beautiful argument works under conditions of reflexivity that are not available when we consider the Brouwer fixed point theorem. The moral of the reflexive domain is that it describes a certain kind of cybernetic environment where every element is also an actor. It is an idealization of the place where action and object are identical. This is an important conceptual domain for cybernetics and needs to be understood more than ever in the present social and political world.

« 7 » **Van Bendegem** also remarks on the amazing behavior of the differential equation $df/dt = f(N - f)$ in relation to its discretization. The discretization behaves chaotically in a complex manner that the differential equation does not display. Important behavior of the discrete system is lost in taking the usual limit. What is the eigenform here? This question requires study. Many people have regarded the chaotic part (which has the *form* of the famous logistic recursion) as something to be studied in its own right, something that is lost when using the differential equation. In that case the full complexity of the logistic recursion is the eigenform, and one can devote one's life to it. No one said that eigenforms were simple. It is what is produced by a certain action. *An eigenform can be what is produced by a certain action*. The invariance is the fact that it is produced by a certain action. A given action produces a determinate result. $Y = F(X)$. You start with X and get Y. Define $T(A) = F(X)$ for any A. Then if $A = Y$ we have $T(Y) = F(X) = Y$. I apologize for not making this fundamental eigenform clear in my essay. Any repeatable producible result is an eigenform. All of the results of science are eigenforms. It is a tautology and it is a *significant tautology*.

« 8 » **Van Bendegem** ends his wonderful criticisms with the injunction to examine the relations between the countable and the uncountable in terms of eigenform. This is a favorite of mine and I cannot resist. Let Ax mean that "x is a member of the collection A." Suppose that we have a method to associate to every member of a collection A a subcollection of A. That is, suppose that for a in A we have F(a) a subset of A. Now define $Cx = \sim F(x)x$. This means that x is a member of C exactly when it is not the case that x is a member of $F(x)$. Could it be that C, being a subset of A, could be of the form $F(z)$ for some z? Let us try. Then $F(z)x = \sim F(x)x$ for any x. So, we let x be z and find that $F(z)z = \sim F(z)z$. We have produced a fixed point for negation ~! But in the world of standard logic and sets it is not allowed that negation should have a fixed point. Therefore, C is not of the form $F(z)$ for any z. This means that the number of subsets of A is bigger than the number of elements of A. If A is countable, then the number of subsets of A is uncountable. Uncountability has arisen from the *denial* of an eigenform! In standard mathematics we have reified the uncountable infinities and then found that they lead to fantastic properties and conundrums. It is a widely accepted mathematical viewpoint (the normal "reality" for nearly all trained mathematician) that arises because we do not allow a certain fixed point. In standard mathematics it is a given that there is no fixed point to the negation operator. I will stop with this remark. I am not saying that we should devalue uncountable infinities because they arise from a denial of a certain form. I am saying that eigenform is always there whether you take it or do not take it. For the mathematicians, there is a belief that the systems produced under these conditions are consistent. If it were to be shown that denying a fixed point for negation would lead to a contradiction, then there would be a shift of very large proportions. I mention these matters to show how "realities" arise for groups of scientific workers. We could make similar examinations in other natural sciences. In this sense of the use of the word reality I do not take it to be forbidden. Each reality is an eigenform with deep internal reflexive structures involving groups of persons acting out the limitations of the form.

« 9 » I thank **Karl Müller** for the many encouraging remarks and would like to say that I am myself very heartened by his concept of endo-science as expressed in his §7. In endo-science we can take a given scientific field and include within it not only the "science" as we come to know it, but also the scientists and their larger relationship in human society. I think that those of us who wanted to become scientists when we were young were keenly interested in endo-science. We devoured biography about the scientists and we wanted to know what it was like to be such a person. There is a popular interest in this dimension. Witness books such as "The Double Helix" by James Watson (1968). And if we keep observing science and the scientists then we shall see how all science is a human endeavor and yet can produce apparently objective and new knowledge about our worlds, perhaps always at the expense of insisting on certain (eigen) forms and denying others. Of course, there is much more in this domain: We want to know. We want to produce new results. We want to find out about their consequences. We want to have responsibility for the consequences. We cannot have everything, but we must not remain in our ivory tower. It is neither safe nor intelligent, nor ultimately fun to do that any more. In my opinion, the most satisfaction comes in reaching for the largest possible observation.

http://constructivist.info/12/3/246.kauffman

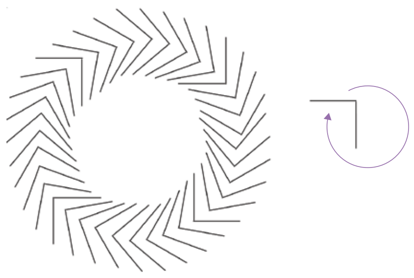

Figure 1 • Circular reentry with circular symmetry.

« 10 » In his §11, **Tom McFarlane** brings the theme back to the fundamental matter of distinction. I agree entirely. A favorite point that I like to make is that the mark of George Spencer Brown in his seminal book *Laws of Form* (Spencer Brown 1969) is self-referential since it is seen to make a distinction and it stands for the (act of) making a distinction. Thus, the mark is an eigenform of itself. In the formalism, we do not have $<< >> = < >$. It is not an eigenform in *that* sense! One needs to understand that the mark $< >$ designates itself without writing extra marks to indicate that self-reference. Doing that act of understanding one arrives at a fact that cannot be expressed without complication, because the very act of expressing it obscures its nature. For example, we could write “$< > P < >$” where “P” stands for the act of pointing. But then in this expression there are two marks, one on each side of the P and the reader must understand that each stands for the same mark.

« 11 » When we write one mark $< >$ and remark that this mark stands for itself, we have accomplished a thought that becomes more complex when it is articulated than it is when it is thought. This is my opinion. The point of this discussion is important for eigenform, for ultimately eigenform occurs in the boundary between understanding and articulation.

« 12 » **McFarlane** points out the beauty and utility of the concept and action of symmetry in eigenforms. This is a deep and wonderful remark. Let us begin to understand symmetry. In the simplest evocation of eigenform, an infinite composition such as $E = <<<<<\ldots>>>>>$ so that $<E> = E$ we have a precise symmetry in the infinite form E that shifts it once inward. This is the same pattern as the symmetry of the integers defined by $F(n) = n + 1$, but note that for the integers F is defined for both positive and negative numbers and the mapping $F: Z \rightarrow Z$ is both 1-1 and onto. Here I take Z to be the set of all integers $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. A corresponding E might look like $EE = \ldots<<<\ldots>>>\ldots$ so that it never stops, inside or outside. This EE does not have a “0-place” but it is symmetrical both inside and outside. Another example of symmetrical eigenform is illustrated in Figure 1 where we have a circularly reentering mark.

« 13 » Once symmetry enters, as it does in the circular eigenform of Figure 1, we find that a number of themes come forth. One theme is that mathematical language is not as expressive as the geometry of the form itself. In this way, the geometry comes forth as a new language in which to express symmetry and relationship. Another example of this tendency is exemplified by the Fibonacci Form $F = <<F>F>$, one of the simplest eigenforms other than the simple reentering mark $J = <J>$. As we know, there is much development of geometry related to the Fibonacci form and its appearance in nature, from the Nautilus sea shell to the sunflower. In all cases the form is clothed in the beautiful symmetry of its enlarged context. We see that symmetry and significant context are closely related, nay inextricably intertwined with each other. **McFarlane** turns to this theme in §8. What he writes points precisely to the role of context and how “deeper” eigenforms will arise that correspond to invariances within a given context. The theme continues into the way physics holds elementary particles in terms of their symmetries and the way as in §7, geometers and topologists continue to look for the deepest and simplest descriptions of the essence of the “objects” of their study. Deeper eigenforms are often the most difficult to express, and can give the most scientific satisfaction when even a partial answer is found.

« 14 » In §4 of his commentary **De Zeeuw** says that “There is no justification for the assumption that some ‘object for our perception’ exists that corresponds to an eigenform.” And he suggests that this is a gap to be filled in the epistemology of eigenform. I do not exactly disagree with this statement. My reaction is this: If there is no justification that some object of our perception exists then it is *just* an eigenform and nothing more. Hallucinations and certain optical illusions fall into this category. Some person might attempt to convince us that our perceptions are populated only with eigenforms and there is no justification for any other kind of existence. But what about the lamp on my desk? Can I justify its existence? I can investigate how it is and how it works and of what it is made. This is a nearly endless investigation. It includes the chemical and physical composition of the matter of the lamp and its electrical properties. It includes the history of its invention, and this leads out into the history of all technology. We can continue, and eventually I find that the story of what that lamp is, is the story of what the whole universe is in which I live and breathe, including all my actions, reactions, thoughts, hopes and fears. This is not a metaphor. It is literally what becomes of the investigation into what any given thing is in one’s world. All that becomes the eigenform of the lamp, indistinguishable from the lamp itself and indistinguishable (except at the cost of denying certain eigenforms) from the observer who in this temporal context is taken to be myself. The quest for the identity of any “ordinary” object such as the lamp on my desk leads outward into an investigation of the entire world of the observer. After all that, the object is not separate from the observer or from that world, but something of great use has been understood. The actions that generate this investigation take us out of the skeletal abstraction and into the whole world of possible and actual relationships. (If the reader is still incredulous that cosmology would enter into the understanding of an ordinary object, consider the fact that in a rotating bucket of water, the liquid will move up along the sides of the bucket. Consider that reverse point of view with a stationary bucket and the whole universe rotating about the bucket. The result must be the same by the relativity of the description. In this way Ernst Mach (Barbour & Pfister 1995) concluded that the effect of the water moving up the sides is due to the gravitational effect of the entire universe in relation to the motion of the bucket. In a universe empty except for the bucket, there could be no such effect.

« 15 » In his §6, De Zeeuw remarks that the initial act of transformation and the initial state that is transformed is all-important for the actuality of an eigenform. I agree fully, and refer to my §§1–3 in this response for my full response on this point. I have placed these paragraphs at the beginning of this response because they apply to everything that is said here in a multitude of ways. We can view the inception of an eigenform as a dynamical invocation of the inception of a distinction. *At first what is acted upon and what is the actor are not differentiated from each other.* Then as time goes on (time itself emerges from the act of distinction) the roles of actor and actant (the one acted upon) begin to be separable. At last there comes a time when actor and actant can be at least partially separated, and one can say "This acts upon that." "I speak and they listen." "The electron hits the screen." Once it is understood that the beginning of such processes is undifferentiated, it becomes easier to understand how certain situations arrive at their distinctions in different orders from what we have been brought up to believe is the norm. For example, in a double-slit experiment we have been taught by Niels Bohr and Werner Heisenberg to say "The electron hits the screen" *only after the observation of an event on the screen has occurred.* In our grammar from childhood, to say that an electron hits the screen is to give that electron existence prior to the time of the meeting with the screen. This is forbidden speech for Bohr and Heisenberg, who would construct an epistemology of eigenforms for quantum mechanics where the invariance only begins at the point of actualization. It is a bold attempt to change the language of action, and in it we see once again how the descriptive worlds of science are built on the construction and observation of eigenforms, and equally on the forbidding or avoiding of other possible eigenforms. Once these contexts are understood, we can speak about action. Before such access to context, the notion of action is fraught with our assumptions derived from the status quo.

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