Authors' Response

The M-N-L Framework: Bringing Radical Constructivist Theories to Daily Teaching Practices

Philip Borg, Dave Hewitt & Ian Jones

> **Upshot** • We seek to address several questions and statements made in the commentaries by elaborating on the four main aspects of the M-N-L framework. Before doing so, we discuss the issue of constructivist teaching in the context of schools. We conclude by hypothesizing on what would be lost in the M-N-L framework by taking constructivism out of the picture.

Teaching from a constructivist stance

« 1 » Over the years, constructivist literature has included debates about the legitimacy of the term "constructivist teaching" (CT). In our target article, we attempted to partly reconcile opposing views by proposing that CT may be attributed to teaching with a sensitivity to constructivist notions. We described teachers' *actions* in the classroom incited from this sensitivity by the Mathematics-Negotiation-Learner (M-N-L) framework that was used by the first author[1] to analyze whether, when, and how CT was taking place in his own mathematics lessons as a school teacher. **Erik Tillema** asks whether it would be better to consider changing the term CT to "teaching from a constructivist stance" (Q1), which is more or less what Martin Simon (1995) refers to as pedagogy from a constructivist perspective. The meaning we give to CT does include teaching from a constructivist stance but it is not only that. Building on the work of Leslie Steffe (e.g., 1991a) we regard CT as also involving teachers' attempts to construct second-order experiential models of their students' knowledge and to let those models affect their own content and pedagogical knowledge. The process of building a two-way road connecting teachers' knowledge and these models of students' knowledge is our concept of "negotiation" in M-N-L.

« 2 » CT is inherently student-centred because it is wholly dependent on whether students are motivated and enabled enough to take an active part in the lesson. It is only by carefully observing students' communication and activities that teachers can build experiential models of their students' conceptual structures. Being student-centred, CT is often confused with progressive methods of teaching like problem-solving, discovery, and inquiry, which espouse a hands-on, active student participation. The overarching philosophy behind such methods is that teachers need to acknowledge that learners have a mind of their own and should allow them to take control of their own learning. Such was the philosophy of John Dewey, who began his campaign for a more active and self-directed style of learning over a century ago. The progressive movement that Dewey (e.g., 1902) advocated calls for teachers to link the knowledge they intend to teach to their learners' "experiential worlds" (Glasersfeld 1990a: 22), something which today may be identified with radical constructivism (RC).

« 3 » In response to **Hugh Gash**'s Q1 about our claim that CT should not be equated with progressive modes of teaching (target article, §1), we clarify that progressive modes of teaching might well have a CT approach but that CT does not necessarily imply a progressive teaching method. Our claim is that traditional teaching methods (which are sometimes carelessly dubbed as "teacher-centred") and CT are not necessarily mutually exclusive. We agree with **Gash** that CT does not translate itself into one particular teaching method and individual teachers may adopt different methods of engaging in CT. In fact, the teacher in our study preferred to use a mixed-methods approach. For example, in the same lesson, one could observe teacher exposition, class discussion, group work, inquiry, and problem-solving activities. If a third-party observer were to enter the classroom and observe the teacher during a teacher exposition episode, that observer could mistakenly deem that as a teacher-centred and pouring-in method of teaching. We attempted to show that even during such instances the teacher was engaging in CT if and when he was sympathetic to constructivist notions of coming to know and strive to negotiate paths between his knowledge and the models he was building of his students' knowledge.

« 4 » The main factor that determines whether teachers engage in or disengage from such a negotiation is their sensitivity to students (SS), in particular to the way students seem to be making sense of the classroom experiences or learning offers (Steinbring 1998). **Barbara Jaworski** Q3 questions how such sensitivity can be seen as a constructivist construct. SS is a teaching characteristic in **Jaworski**'s (1994) "teaching triad," a descriptive model derived from studying the way a classroom teacher, Clare, went about her daily business of teaching mathematics. Clare exhibited SS by her efforts to become aware of her students' conceptual structures, thinking styles, and tendencies, but not only that. Her SS was particularly evident when she was observed striving to make her learners feel respected, included, and cared for in such a way that they felt at ease communicating to her what they were thinking. In the M-N-L framework SS plays a very important role. If teachers do not have SS or choose to disregard it, they cannot create an atmosphere of interaction and communication of ideas with the students and the "negotiation" paths from "mathematics" to the learner will be obstructed, as we attempted to show by Protocols 5 and 6.

« 5 » In our understanding of CT, teachers need to "empathize" with their students before, during, and after interacting with them, especially when they seek to build second-order experiential models of students' knowledge. This "empathy" needs to be affective before it can be conceptual. In Protocol 1 (target article, §22) the teacher *sensed* that Omar was frustrated (Omar had let out an exasperated sigh) because he was getting an error message from the software. The teacher *acted* on that sensitivity and asked Omar, "What's the matter there, mate?" His use of the word "mate" or "pal" (in Maltese, "*xbin*," pronounced "*shbeen*") was an attempt to establish an affable, same-level communication which seemed to make Omar comfortable to talk about his difficulty in achieving his goal. Without establishing this friendly, caring rapport it would

1| The first author was the teacher-researcher in this study. He will henceforth be referred to as "the teacher

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have been unlikely for the teacher to form a model of the mathematics of the student (MOS) and identify the mismatch between this MOS and the "mathematics" coded in the software (which was the teacher's mathematics – the mathematics he wanted to teach). During this protocol the teacher

a was *affectively* sensitive to Omar's frustration/exasperation,
b acted on his affective sensitivity (started the conversation in a casual, friendly manner),
c was *cognitively* sensitive to Omar's mathematics, and
d acted on his cognitive sensitivity by asking questions to enable him to construct a model of Omar's conceptual structures.

So to answer **Jaworski**'s question, rather than being a constructivist construct per se, SS is a necessary characteristic for teachers to engage in what we are defining as CT. Whether teachers engage in CT depends on whether they decide to *act* on that SS. It also depends on their outlook towards the subject matter, in our case mathematics. In the following section, we elaborate on our definition of "mathematics," mostly in response to **Steffe**'s §2.

"Mathematics"

« 6 » Since we defined "mathematics" as the consensual domain (CD) existing among a group of persons, it seems appropriate to elaborate on our understanding of CD and the verb "existing," given **Steffe**'s (§2) objections to how we use these terms. Ernst von Glasersfeld argues that based on their different, subjective views of their experiential worlds people

> "can build up a consensus in certain areas of their subjective worlds. Such areas of relative agreement are called 'consensual domains,' and one of the oldest in the Western world is the consensual domain of numbers. The certainty of mathematical 'facts' springs from mathematicians' observance of agreed-on ways of operating, not from the nature of an objective universe." (Glasersfeld 1991a: xvi)

« 7 » Anyone who has constructed and developed some mathematical *concept* is a kind of "mathematician." Such concepts include those that do not make use of agreed-on conventions, like the concept of quantity (you don't have to know about numbers to make decisions about which tree is taller, which bag is heavier, which handful of sweets contains more…). Therefore, there is a global mathematical CD among members of the human race: any group of communicating persons would agree that a 20 m building is higher than a 10 m one, even if they never learnt about the quantities 20 and 10. However, within this CD of "mathematicians," people have invented, used, and passed on mathematical *conventions* with which the sense-making of their experiential worlds became more efficient and viable. A quick glance around us shows the extent to which these conventions are applied to enable us to cooperate in our search for explanations of our experiential world. Philip Steedman (1991: 7) says that mathematics is "a social creation which changes with time and circumstances." But as long as the mathematical concepts and conventions serve their purpose to help humans cope with their experiential worlds, they prevail, and for that purpose it is deemed necessary that they be included in school curricula in the form of branches, topics, and sub-topics. It is then up to teachers to *interpret* the "mathematics" written in the curriculum and re-present it through actions in the classroom, with the aim of bringing it "within reach" of their students, to acquaint them with conventions and concepts which may prove viable to explain their experiential world. **John Richards**'s (§2) distinction between *inquiry math* and *journal math* is very pertinent here. Teachers need to get to the heart of the topics in the *school math* written in their syllabi and convert it to *inquiry math* by giving their students the opportunity to be "actively engaged" (Richards 1991: 38) in constructing their own meanings for themselves while participating in mathematical discussions or problem-solving activities. In doing so, teachers seek to find ways to extend and expand the CD within which their students think and operate. This is achieved if students are willing to *interpret* the mathematical re-presentations of the teacher and internalize or rebut their teachers' mathematics and re-present their own mathematics (MOS).

« 8 » "Mathematics" is thus something that is interpreted and this, we believe, makes it something that "exists." George Berkeley's famous Latin-English coined dictum, "*esse is percipi*" (to exist is to be perceived) warrants the claim of "existence" of entities by virtue of their being perceptible (Berkeley 1949: part 1, §3). Not unlike Berkeley, we claim the existence of "mathematics" in the CD of mathematically communicating individuals to be warranted by its *interpretive* property because interpretation is a key aspect of perception. For perception to occur, minds need to interpret what eyes, ears, and hands see, hear, and feel.

Mathematics-to-Learner Negotiation

« 9 » In their effort to place "mathematics" at the disposal of their students, teachers need to make decisions as to what part of that mathematics is fit for their students (MFS) and how to go about interacting with students. This requires an anticipation of the teacher-learner interaction where teachers ask questions like

- What would the students think of the mathematical problem?
- What are the possible experiences of the students that I may tap into in order to help them make sense of what I am re-presenting?
- What possible difficulties may the students have when dealing with this concept?

« 10 » In order to start answering these questions, teachers need to concern themselves with how their students come to know (cf. **McCloughlin**'s Q1) and rely on models they build of the current or "similar" students' ways of coming to know. Teachers use this knowledge to choose how to interact with the students and we gave a number of examples of how this could be done (target article, §5). **Jaworski** (Q1) asked us to elaborate on how teachers' constructivist beliefs would be reflected in the way they approach this task. If teachers assume an RC stance, they need to see how their beliefs about knowledge and cognition are reflected in their thoughts and actions. Their belief that knowledge is actively constructed, not transmitted and received, should keep RC teachers from trying to force their methods onto the students. Rather, they seek to *learn* about students' reasoning and identify pathways between that reasoning and

MFS. The negotiation of those pathways is the orientation of students' ways of thinking. This comes with RC teachers' belief that knowledge serves students to make sense of their experiential worlds rather than to gain access to an absolute reality. Rather than presenting "mathematics" as a collection of "truths" about the universe, RC teachers will present it as a creation which can help humans explain and possibly predict experiential phenomena. They will try to motivate their learners to enrich their mathematics (MOS) by challenging them with situations where the students would feel the need to build or restructure their knowledge, in the pursuit of a more viable idea or way of reasoning. Furthermore, RC teachers hold that "knowing" is a state that can only be achieved when learners actively construct ideas in a process we generally refer to as "learning." We agree with Steffe (§10) that RC is a theory of knowing but, being such, it is necessarily a theory of learning. In fact, Glasersfeld's (1990a) first RC principle is more about the process of coming to know (learning) than it is about the nature or purpose of knowledge itself.

« 11 » RC teachers, therefore, make it their business to inquire about their students' current knowledge, in our case, MOS. Sometimes this is done "by posing problems, asking questions, [and] discussing" (Richards §6), as the teacher in our research was doing in most of the protocols. Perhaps this is why Steffe (§3) regards the protocols as "couched almost exclusively on the teacher's side of the interaction." Quite the contrary, we argue that Protocols 1–4 show the teacher asking questions in order to give students the opportunity to express themselves in the discussion so that he could build models of their MOS. In particular, Protocol 4 shows how Dan was given the opportunity to re-present, through verbal and bodily expressions, an analogy between an equality and a barbell with weights. As an example of student interaction (Steffe Q1), we present the contributions made by the students in all the protocols, including Protocols 5–6, where the teacher was observed not to engage in what we define as CT. The students interacted with the teacher by *interpreting* the teacher's expressions (utterances, gestures, drawings, demonstrations, etc.) and *acting* on those interpretations by re-presenting their own mathematical conjectures. Apart from expressions that one would expect in classroom discussions (re-presenting conceptual structures through verbal statements and questions, tone of voice, facial expressions, gestures, role-playing, etc.) the students had the opportunity to make re-presentations through drawings, symbol-writing, and activities on the interactive board, on their computers, and on paper. This is discussed in more detail in Borg & Hewitt (2015). Since our article was concerned with the analysis of the teacher's actions in the classroom with the help of the M-N-L framework we did not discuss student-student interactions and student-computer interactions unless the teacher was taking part in those interactions. This might have falsely depicted a picture of a classroom where the teacher did "not relinquish control [and] seemed to be centrally involved" (Steffe §6). Actually, the second half of each lesson was dedicated to letting students work in pairs on their computers on goal-oriented tasks and the teacher was only involved if students asked for assistance or if he *sensed* that his assistance might be required, as was the case in Protocol 1. This might address Steffe's concern expressed in Q2.

« 12 » Furthermore, to reply to Gash's Q2, the use of M-N-L to analyze CT is only a part of a larger research project that the first author is currently undertaking[2] in which part of the video analysis does concentrate on students' learning and interactions, particularly on whether and how students participated in cooperative learning, especially when they were working on their own in pairs. Nevertheless, what happened on the "Learner" end of M-N-L is pertinent to the overall question of CT and we are going to clarify some issues about this in the next section.

2| This is a PhD research project under the supervision of the second and third authors. In response to Gash's Q4, the lesson video analysis in this research project is indeed serving as a means of professional development for the first author with respect to his attempts to implement RC in his daily practices as a teacher, where his co-authors are indispensable mentors in giving meaning to the video observations.

Learner

« 13 » One of the most important features of M-N-L is that it characterizes CT as the type of teaching stance in which teachers learn from their students. This is done by encouraging students to *reflect* on a learning offer and contribute to class discussions or goal-oriented activities (in our case, these were set with the help of the software Grid Algebra). Tillema's Q2 refers to an important issue: how can constructivist (mathematics) teachers act on the belief that the meanings inferred in classroom communications do not have an ontological status outside of the ones that the communicating individuals confer on them? Teachers acknowledging this constructivist notion and who are aware that part of their duty as teachers is to negotiate their "mathematics" to the learners, need to listen to and immerse themselves in these communications and, if necessary, *orient* their students' thinking by presenting them with a "situation in which the students' network of explanatory concepts clearly turns out to be unsatisfactory" (Glasersfeld 2001: 10). This requires care that the teacher is not "merely 'tricking' the student to arrive at the same conception as the teacher" (McCloughlin §10). Teachers may contribute with their own meanings in the discussion and negotiate pathways between the "mathematics" in the syllabus and that of the students by creating "perturbations" in the minds of the students, the settlement of which is bound to result in students learning new concepts or modifying old ones. In addition, constructivist teachers must be open to letting their pedagogical and subject-matter knowledge be "perturbed" and to learn from the meanings that their students assign to the concepts being discussed.

« 14 » In order for teachers to create a CD in the classroom community (which includes themselves) they must seek to establish a state of taken-to-be-shared meanings. This means that when a student makes a significant contribution to the classroom interactions affecting the CD, the teacher either challenges or amplifies the contribution with the help of that student and by encouraging other students to join in the interaction. We could not include extensive protocols in the target article but to address Steffe's Q3, an individual student's contribution to classroom discussions and activities was indeed imputed to other members of the classroom community, but only after further discussion enabled the teacher to create models of students' ways of thinking in which students were observed to share the reasoning behind that contribution. In the case where students disagreed, the teacher challenged and encouraged students to reflect and discuss further, aiming to bring out the viability and unviability of the different perspectives and to help students adopt the explanation that was proved viable in that particular situation.

« 15 » Tillema (Q4) asks about distinguishing between a situation where students achieve a state of taken-to-be-shared meanings and agreed-upon concepts without making fundamental changes in their conceptual structures and a situation where such a state is achieved through the accommodations of new ideas in students' mental schemas. The distinction was not addressed explicitly in our analysis of the protocols, mostly due to our focus on the teacher's actions vis-à-vis CT. In Protocol 3, for instance, the students seemed to accept Dan's suggestion that the series 1, 2, 3,… could be seen as the one-times table. Given that they were all supposedly familiar with multiplication tables, we do not think that students had fundamentally changed their mathematics. Still, the possibility that they became aware of yet another way of looking at the positive integer series was an important foundation for what the teacher intended to introduce afterwards (the other multiplication tables and the relationship between numbers in different tables). There are other instances in the study, which we do not have the space to delve into here, where classroom interactions appeared to lead to deep and fundamental changes in the way students thought about some mathematical conventions. One such episode happened when students were presented with the statement $3 = 2 + 1$, where some students were adamant it was written in the wrong way since they were accustomed to seeing the equality symbol written at the end of a calculation and preceding the answer. It took more than one lesson episode to achieve a CD about the balancing property of the equality symbol, one of them being the episode from which Protocol 4 was taken.

Learner-to-Mathematics Negotiation

« 16 » What constructivist mathematics teachers do with what they observe and perceive regarding students' dealings with the classroom experiences is the second negotiation "road" between learner and "mathematics." Teachers build second-order experiential models of MOS that they use to revisit MFS. Our definition of MOS is narrower than that of Steffe (e.g., 1991a), which encompasses any student construction "that could be thought of as mathematical simply because they are human beings" (Steffe §8). As used in the M-N-L framework, MOS is the "mathematics" as described in §§9–11 above. Nevertheless, as Jerome Proulx (§1) states, MOS lies in the eyes of observers (teachers or researchers) who interpret what they observe according to their own experiences and the mathematical constructs they have construed for themselves.

« 17 » Jaworski (§4) associates our notion of teachers building second-order models of MOS, to Aaron Cicourel's concept of observing a social relation. We do not completely agree with this association. Cicourel (1973) was building on Alfred Schütz (1964), who distinguishes between someone observing interactions between actors in a *We-relation* and someone observing interactions between actors in a *They-relation*. In the former, the observer observes an interaction in which she is taking part. In the latter, the observer is not an actor in the interaction. Writing from the perspective of an observer, Schütz (1964: 55) says that "whereas my experience of a fellow-man [sic] in the We-relation is continuously modified and enriched by the experiences shared by us, this is not the case in the They-relation." Cicourel (1973), as cited by Jaworski (1994), is talking about observing a They-relation. In our article the teacher was involved in observing both types of relations. When acting as a teacher he observed a We-relation when he was involved in an interaction with the students and he also observed a They-relation when he was perceiving an interaction going on between the students in which he was not involved. When he was acting as a researcher, analyzing video recordings of lessons and computer activities, he was always observing a They-relation, even when he was involved in an interaction, because he could not somehow repeat the exact train of thoughts that went through his mind during the face-to-face encounter with the students.

« 18 » Schütz distinguishes a *pure* We-relation from a *concrete* We-relation. In the first, observers merely acknowledge that someone is "in front" of them but do not know if or how that person is comprehending them. In the latter, observers know how that someone is oriented toward them. To give an example, teachers may be aware that the students before them may or may not be paying attention to them (pure We-relation) but they can gain information about whether and how the students are comprehending them through "concrete manifestations of [the students'] subjective experiences" (Schütz 1964: 27) (concrete We-relation). Analyzing from an RC stance, the first author was not assuming the certainty Schütz (1964) seems to propose when claiming that in a concrete We-relation the observer could gain knowledge about how the other person is comprehending him/her. Whenever the teacher constructed models of MOS, both as a teacher and as a researcher, these could only be second-order models. Jaworski (Q2) asks how this position aligns with constructivism. It is precisely the assumption of uncertainty about students' conceptual structures that is key to the application of RC in the M-N-L framework. By talking about a second-order model formation of MOS (target article §5), we are implying that:

- The meaning students give to teacher-student interactions are subjective, individualistic, and unique. They might share a mathematical CD between them and with the teacher but that does not make their mathematical interpretations and re-presentations less unique.
- Students cannot share their understandings with the teacher (or with one another). As Glasersfeld (1995: 48) says, "the expression 'shared meaning' is misleading" because people can only talk about or make external re-presentations of those meanings. Teachers can only build a model of what their students are saying or manifesting by referring to associations of their own experiences with those words or actions.

« 19 » The RC stance of the M-N-L framework does not stop with teachers' constructions of models of MOS. Constructivist teachers allow their "mathematics" to be affected and often perturbed by what they learn about MOS (target article, §5f). The settlement of this perturbation results in a reviewed and sometimes modified MFS. In Q3, Tillema asks about the intended MFS and how this relates to the teacher's model of MOS. This is a difficult question to answer in a few words because there were quite a number of mathematical meanings that the teacher sought to discuss with the students during the course of the twenty double lessons spanning the whole scholastic year. Broadly speaking, the aim of the lessons was to help students familiarize themselves with formal algebraic notation with the help of Grid Algebra. This involved concepts like unknowns, variables, equalities, and order of operations. Since Grid Algebra uses the multiplication grid as a basis for its users to construct meanings about these concepts, students needed to get acquainted with the way it worked, particularly how specific numbers were designated to specific cells and how numbers or expressions in those cells related to others. In each lesson the teacher was repeatedly toing and froing between learner and "mathematics" by detecting or creating links between MOS and MFS. The few instances where the teacher was observed not to take MOS into consideration in his communication of MFS was taken as a failure on the part of the teacher to engage in CT, at least for that episode where the teacher seemed to disregard his sensitivity to students' individual constructions of MOS. We agree with Gash's Q3 that teachers need to have the flexibility to postpone their actions on a student's contribution to the classroom discourse. His suggestion that when they need to do so teachers should acknowledge that student's intention to interact and show him/her that they would follow it up later is both pragmatic and constructivist. If teachers do this, they would still be showing their sensitivity to what students might be constructing and their intention to act on that sensitivity.

« 20 » The use of "negotiation" or "negotiated meanings" in constructivist literature is not necessarily used in the way we have described above and in the target article. Thomas McCloughlin (Q2) asked why we did not refer to the CAME project. Mundher Adhami, David Johnson & Michael Shayer's (1995) use of "negotiation" was derived from the work of Jörg Voigt (1994), whose concept of "negotiation" was the settlement of academic "conflicts" between teachers and students arising principally from their different points of view. Our use of "negotiation" is somewhat different. In M-N-L, teachers negotiate paths between what lies in the curriculum and what is already constructed in the learners' minds. Sometimes this is done by simply associating or assimilating MOS with MFS. At other times teachers need to adapt MFS while keeping in mind that they can never abandon the "mathematics" in the curriculum that they are duty-bound to help their students learn.

Abandoning the constructivist stance?

« 21 » Jaworski (Q4) questioned what would be lost in M-N-L if it abandoned constructivism as an overarching frame. Since M-N-L was devised from an RC stance we will attempt to answer this question with the assumption that RC was absent from this framework. If this were the case, the meaning of "mathematics" would be very different. It would simply be a body of knowledge that exists in its own right, irrespective of whether it was interpreted or re-presented. Teachers would not need to care about the individual experiences and hence diverse interpretations of the students. They would not need to learn anything about students' knowledge construction or, in our case, MOS. Their MFS would simply be the next topic in the syllabus and if students do not seem to be making sense of it they would assign more drill work until somehow the students started getting the correct answers. Teachers would not have to bother about understanding as long as students "performed" well. The aim of teaching would not be to help students construct meanings but for them to perform competently. Glasersfeld (1995) says that this is training rather than teaching.

« 22 » If RC were taken out of the picture, the whole rationale behind teachers' having to think of ways to negotiate between mathematics and learner would vanish. Why would teachers need to anticipate didactic processes according to models they create of experiences of the current or similar students (target article, §28) if mathematical concepts exist independently of experience? Why would teachers need to learn about their students, specifically about their construction of MOS if this had no bearing on what "mathematics" were on offer? The two-way negotiation road where teachers are also learners and learners are also teachers would be replaced by a conveyor-belt system that transported a body of a priori knowledge from teachers' mental databases to the students. Consequently, there would not be any reason for teachers to encourage reflection on the part of the learners, if all learners had to do were to accept what the knowledge conveyor belt brought their way. Teachers would not be the intermediary agents between "mathematics" and learners. Their role would simply be to place mathematical facts and techniques on the teacher side of the knowledge conveyor belt. The "negotiation" in M-*N*-L would not be a negotiation at all.

« 23 » We hope this re-affirms the indispensable constructivist stance of the M-N-L framework as much as we hope that this framework serves teachers and researchers to bring RC theory to school teaching practice.

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