to apply ideas "generated" in the lesson to problems in the world we experience.

c *Cognitive conflict* – challenging certainties, which can be thinking about a problem in a way that challenges prior knowledge.

d *Metacognition* – the explicit cognizing of students' constructions, which can be reflecting on thinking and articulating approaches to solving a problem.

e *Concrete preparation* of the cognitive "space" to determine "readiness" and which can be introducing a problem and helping with any new vocabulary or ways of doing.

« 9 » Negotiation is a feature of democratic and non-authoritarian personal interactions, as is metacognition. Social construction also involves negotiation and cognitive conflict can only be resolved satisfactorily through negotiation. Therefore, it is difficult to understand the decision neither to refer to the CAME Project, nor independently refer to metacognition or cognitive conflict. Therefore, this gives rise to a second question: **Why did Borg et al. not refer to or consider the CAME Project (Adhami, Johnson & Shayer 1995; Adhami, Robertson & Shayer 2004; Adhami, Shayer & Twiss 2005), or the wider implications of the cognitive acceleration paradigm? (Q2)**

« 10 » In the learning sciences, negotiation is best thought of as the means of seeking the student input into their own learning to arrive at a shared understanding. However, in the constructivist paradigm, it is not merely "tricking" the student to arrive at the same conception as the teacher by the artistry of teaching. In the constructivist paradigm, the teacher devolves their control by declaiming the ultimate authority of the subject content material itself, and thus the goal of mathematics education is not to acquire the mind of the teacher / mathematician, but rather develop the constructions of the student to be what they can be. The CAME project uses negotiation to assist learning in one or more, but at times all of the methodologies listed above. The thing is, negotiation, if authentic, does not try to steer a course to a direct goal known or pursued by one party to the negotiation. In constructivist learning, and teaching, negotiation will not be trivial, or inauthentic, but demand an openness that goes beyond mere sensitivity and makes the teacher as accountable to the learning contract as the student.

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Mathematical Observers Observing Mathematics

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> **Upshot** • Suggestions are made for ways of taking advantage of Borg et al.'s reference to the notion of observer for data analysis in mathematics education research.

"Thus, if someone claims to know algebra – that is, to be an algebraist – we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim." (Maturana 1988a: 4f)

« 1 » In order to ground their idea of Mathematics of the Students (MOS), in their target article Philip Borg et al. make use of second-order models and the notion of observers to discuss issues of data analysis. In this commentary, I suggest that these ideas can be further deepened by making reference to Humberto Maturana's theory of the observer. In Proulx (2014) I argue that some aspect of Maturana's observer[3] could lead us beyond, and even question, Leslie Steffe's use of second-order models to describe the mathematics in students' minds. From an observer's point of view, MOS does not "exist" in itself in the students' minds, but emerges and is distinguished by observers through the act of observing. Therefore, MOS lies in the eye of the observer, and what is recognized as mathematics is what the observer recognizes as mathematics on the basis of her own experiences that she understands as mathematical.

« 2 » Henceforth, we can re-use Borg et al.'s views of what the curriculum is for conceptualizing MOS, where MOS could be seen as something that observers have construed for themselves:

"RC teachers, however, believe that no such thing [i.e., curriculum/body of *a priori* mathematical knowledge] exists and that the curricular topics they are required to teach are actually knowledge they (the teachers) have construed for themselves." (§3)

« 3 » This view of MOS through the observer has quite some potential at the epistemological level for mathematics education research in relation to data analysis, as I already claim in a previous commentary (Proulx 2014). Here, I build on this commentary and re-insist and extend some of these ideas by adding concrete examples taken from my studies.

Observing possibilities

« 4 » In Maturana's theory of the observer (e.g., Maturana 1987, 1988a, 1988b), the observer is central to any *account* of any given phenomenon, for "everything said is said by an observer to another observer that could be himself or herself" (Maturana 1988b: 27). As explained in Maheux & Proulx (2015) and Proulx (2014), even if, as researchers in general, we may not believe that "the phenomenon being observed" is a fact independent of the observer and that it can be decontextualized from the observational act, we often take this position implicitly by how we report our findings. As Richard Barwell (2009) explains, even if we agree that we cannot account for what "really" happens, research is still being reported (and maybe even conceived) as if this were the case. This is, for example, the case when

3 | I do not pretend to represent, make use or apply Maturana's ideas of the observer here. The issues I raise have been inspired and occasioned in relation to his work. Thus, there is more in his writing than what I offer here, but also less. But, nonetheless, his writing has made these distinctions possible for me as a researcher.

it is suggested that a phenomenon analyzed from various perspectives "generates a *deeper* or *more comprising understanding* of the [observed] phenomenon" (Bikner-Ahsbahs & Prediger 2006: 56, my emphasis; see also Prediger, Bikner-Ahsbahs & Arzarello 2008). Or when researchers claim that the complexity of, for example, the classroom, cannot be *understood* only from one perspective (see, e.g., Even & Schwarz 2003). This position entails an assumption that "the phenomenon being observed" is a fact independent of the observer and can be decontextualized from the observational act. Therefore, not only can we look at the same data or the same situation from various perspectives, but we can also aspire to an "accurate account" of it.

« 5 » Through being attentive to Barwell's critique, the issue of "accurate account" is meaningless if the researcher holds an epistemological position where one does not *describe* what is being observed, but *constructs* one's own account of one's own perceptions. The adequacy of mathematical actions and strategies is not linked to some allegedly objective referent, but to the eye of the observer who *assesses* it on the basis of her own set of criteria: analyzing data rests no longer in their "truth" or validity, but in what they offer to us and to others.

« 6 » This observer's point of view transforms assertions about what are seen as research findings or MOS, and what can be learned from them. It proposes a transformation of view, not one focused on a state of affairs ("it is" *versus* "it is not") but toward the possibilities, the potential of these actions, to where they lead. In that sense, it is aligned to the mathematician Dave Henderson's view of mathematical correctness:

> "I relate correctness to the goal by saying that something is correct to the extent it moves an individual or group of individuals in the direction of an expanded understanding and perception of reality. […] How do we view mathematical arguments? When do we call an argument good? When do we consider it convincing? – When we're convinced! – Right? – When the argument causes us to see something we hadn't seen before. We can follow a logical argument step by step and agree with each step but still not be satisfied. We want more. We want to perceive something." (Henderson 1981: 13)

« 7 » This view is oriented toward what can be made possible by mathematical actions, or Borg et al.'s MOS, of where and what it can lead to, where it extends; rather than a focus on what it *is*, a state of affairs of what one does or does not know. The observer offers thus an analysis that develops understandings of MOS in terms of their potential, their mathematical possibilities and extensions.

« 8 » With this positioning, the stakes of analyzing data rest no longer in the "truth" or validity of students' mathematics, but in what they offer to oneself and one another. Thus, the approach engaged in for data analysis implies the *necessity* to move away from questions about mathematical knowledge or knowing and focuses on students' actions *for imagining possibilities for mathematics education*, for seeing extensions, rather than arguing for or against taken-as-given practices, activities, tools, and so forth. Following Simon Jarvis's (2004) idea of speculative thinking, the intention is to imagine possibilities, to draw them out by analyzing students' mathematics.

« 9 » Even if it prevents us from making direct assumptions about what students might know or not know – as if they were holding knowledge one way or another – studying students' mathematical actions makes it possible to make sense of these propositions as diverse ways to *approach* and *go about* students' mathematical activity or MOS. That is, regardless of the "trustworthiness" of the students' oral account of their thinking processes, or the possible relation of the strategies observed with fixed/preexisting forms of knowing that one would recognize and relate to, these strategies can be discussed in terms of *action possibilities*.

« 10 » As a way of illustrating these ideas about data analysis in terms of mathematical possibilities, I offer here an analysis of two examples taken from my studies on mathematical problem-solving.

Example 1: Operations on functions

« 11 » In one study, Grade-11 high-school students (15–16 years old) had to solve graphically usual tasks about operation on functions (Proulx 2015a). The graph of two (or three) functions was represented in the Cartesian plan on the whiteboard, and students had 15–20 seconds to operate on these functions and then draw their response on a blank sheet with a Cartesian graph on it (with the line $y=x$ drawn on it as a referent). Figure 1a illustrates a task where students had to add functions f and g, and Figure 1b displays one of the strategies students developed to solve it. To solve the $f+g$ task, students paid attention to the following points:

- where f cuts the x-axis (x-intercept), resulting in an image-length in g (because the image-length in f is 0);
- where f and g cross each other and have the same image-length, resulting in an image double the value of where they intersect;
- where f and g cut across the y-axis (y-intercept), resulting in a similar process as in (2);
- where g cuts the x-axis as in (1).

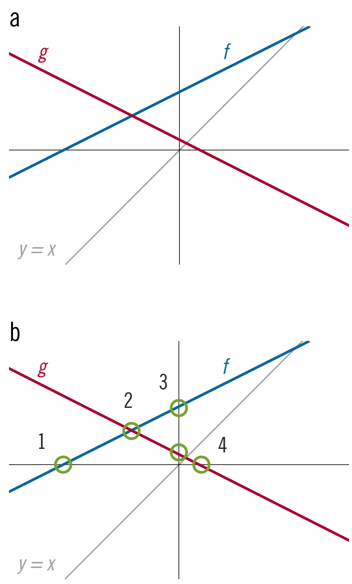

Figure 1 • (a) The $f+g$ task; (b) The strategy focused on significant points to solve it.

« 12 » This strategy, i.e., focusing on significant points that enable one to determine where the line/function is, has potential, e.g., in relation to an extension toward multiplication of functions. Indeed, paying attention to points 1 and 4 permits one to evaluate the general shape of image-length for an x smaller than that at point 1: with negative values of f multiplied with positive images of

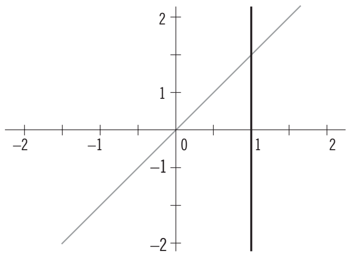

Figure 2 • The line $x=1$ given as the answer to the system "$y=x$ and $y=-x+2$".

g giving negative values in its multiplication; the same applies to images with an x bigger than the one of point 4. For values in x situated between points 1 and 4, the multiplication of images gives a positive value, leading one to see the quadratic function (second degree) created by the multiplication of two linear functions (first degree). Through the use of significant points this strategy reveals graphical aspects related to these functions, and this can also be linked to the study of zeros of functions and inflections points in calculus (e.g., here obtaining [+|−|+]).

Example 2: Systems of linear equations

« 13 » In another experiment, adult solvers (high school teachers) were given 15–20 seconds to solve a number of usual tasks on systems of equations, given algebraically on the board, and then to draw their response on a blank sheet with a Cartesian graph on it (also with the line $y=x$ drawn as a referent) (for more details see Proulx 2015b). In the case of solving the following system of equations "$y=x$ and $y=-x+2$," Figure 2 shows the answer given by one participant, the line $x=1$.

« 14 » This participant drew the vertical line, that is $x=1$, explaining that he did not have enough time to find the value of y, but that the solution had to be on this line because when replacing $x=1$ in each equation, it gave the same value. Note that the substitution of $x=1$ in the equations directly gives the value in y (equations being of the form $y=\mathrm{m}x+\mathrm{b}$). However, in his algebraic manipulations to find the value for x, the emphasis is on finding a common x that gives the same answer ($x=?$ and $-x+2=?$) and not on finding the value for y even if *it is* the same value. But in his strategy, both were done/seen separately.

« 15 » Even if incomplete, this strategy introduces many mathematical possibilities and extensions. The line $x=1$ permits representation of all possibilities for y to solve the system, even if only one value will be able to satisfy both equations simultaneously. In addition, this line $x=1$ represents a family of solutions to the system of parametrical equations "$y=\mathrm{x}+k$ and $y=-x+(2+k)$," leading to the study of parameters with $k=0$ for the parameter of this system that has (1, 1) as a solution. Also, this "omitting" to pay attention to y highlights the interesting obvious fact that the value for $x=1$ for both equations is the value for y, thus also working on the value in y because the value in y needs to satisfy both lines of the system of equations. Finally, the intersection of $x=1$ and the referent line $y=x$ is exactly where the solution point is situated. This makes it possible to insist that the solution *is part of* both lines of the system; a fact that is often overlooked when solving algebraic tasks.

Conclusion

« 16 » As the above analysis shows, the orientation taken toward MOS aims at bringing forth its potential and possibilities in a flexible and developing form rather than as a state of affairs as if it were static or a fixed form of knowledge. To consider it static would entail that mathematics or MOS exists in itself in the minds of students, a position contrary to that which I have developed here. Rather, inspired by Maturana's theory of the observer, I argue that MOS emerges in the eye of the mathematical observer (when) observing mathematics.

« 17 » As mentioned in Proulx (2014), the observer transforms the assertions about what are seen as findings and what is learned from them: pointing to potential extensions, suggesting what could be created by it and inviting us to think in alternative or even "futuristic" ways. Thus when we speak of mathematical possibilities, it is always from the point of view of the observer, in what this observer sees as mathematically possible.

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