tivism, there would potentially be problems in trying to unify Rezat & Sträßer's model with the M-N-L model, which is claimed to be constructivist. Here, theories or theoretical constructs that are plausible in their own right start to give problems when juxtaposed. This is one of the reasons why it is important to be clear and specific as to what aspects of a broader theory pertain to a particular construct.

« 7 » I will end with another question for the authors: **Given that M-N-L seems to be an effective model in theorizing the negotiation of the teacher between mathematics and students, with sensitivity to students (albeit lacking specific attention to the Grid Algebra software), but with dubious links to constructivist theory, what would be lost by abandoning constructivism as an overarching frame? (Q4)**

Barbara Jaworski is Professor of Mathematics Education in the Mathematics Education Centre at Loughborough University, UK. She was formerly Professor of Mathematics Education at the University of Agder, Norway, and before that a Reader at the University of Oxford. She has a career that spans mathematics teaching at secondary level and first-year university level, teacher education at secondary level, and teaching and supervision of doctoral students. Her research has been mainly into the development of mathematics teaching through research and through partnerships between teachers and teacher educators or didacticians at a range of levels. She was for six years Editor in Chief of the Journal of Mathematics Teacher Education and, for three years, President of PME, the International Group for the Psychology of Mathematics Education.

Received: 9 October 2016
Accepted: 18 October 2016

Toward a Model of Constructivist Mathematics Teaching

Leslie P. Steffe
University of Georgia, USA
lsteffe/at/uga.edu

> Upshot • My commentary has two general goals. First, I investigate how basic principles of radical constructivism might be used in constructing models of mathematics teaching. Toward that end, I found that I was not in complete intersubjective agreement with Borg et al.'s use of some basic terms. Second, I explore what mathematics teaching might look like in what is construed as constructivist mathematics teaching. Toward that end, I comment on Borg et al.'s use of "negotiation" and explain how a constructivist teacher can establish experiential models of students' mathematics.

« 1 » Philip Borg, Dave Hewitt and Ian Jones have chosen to address what has become a pronounced problem in outcome-based mathematics education, and that is the disconnection between the mathematics curriculum and students' mathematics. The approach the authors take to gaining insight into this problem is from the perspective of a practising mathematics teacher who attempts to use principles of radical constructivism (RC) in developing a guiding framework for engaging in constructivist mathematics teaching. The framework is based on the authors' meaning of several key terms, some of which I find problematic. So rather than discuss their framework per se, I focus on their interpretation of some of the key terms and consequences following on from the interpretations in order to engage in a discussion of constructivist mathematics teaching.

"Mathematics" and consensual domains

« 2 » The first key term, "mathematics," is taken as a consensual domain of mathematical concepts and skills existing among a group of persons. Because Ernst von Glasersfeld (1990b) spent considerable time interpreting Maturana, I once asked him how he regarded Maturana's idea of a consensual domain. His explanation was that,

> "A consensual domain is established when the individuals of a group adjust their actions and reactions to achieve the degree of compatibility necessary for cooperation. This involves the use of language and the adjustments and mutual adaptations of individual meanings to allow effective interaction and cooperation." (Personal communication)

By using the term "existing" in their account of "mathematics" as a consensual domain, the authors seem to regard "mathematics" as "out there" in-between or among the individuals of a group, whereas von Glasersfeld focused on mutual adaptations of individual meanings to allow effective interaction and cooperation. Positing the "existence" of a consensual domain of mathematical concepts and skills in-between the participants in an interaction is not inappropriate if "existence" is taken from the point of view of an observer who observes cooperating individuals in effective interaction. That is, it takes an observer to interpret the observed interaction and, on that basis, construct what she construes as consensual mathematical concepts and skills "existing" among the participants in an interaction. The observer may well be one or more of the individuals involved in the interaction who assumes the perspective of a second-order observer, which is included in Maturana's account of an observer.[1] In a case where the observer is a teacher, the teacher may construct what she construes as mathematics of students involved in the interaction that she imputes to the involved students. She may also decide where she might choose to take the students in subsequent interaction, which is what the authors refer to as the mathematics for the involved students.

Interaction

« 3 » I found the protocols and the discussion surrounding them consistent with the authors' account of interaction, which is couched almost exclusively on the teacher's side of the interaction, where the students'

1 | "Everything said is said by an observer to another observer, who can be himself or herself" (Maturana 1978: 31).

role is interpreting the teacher's utterances, etc. The authors also consider the principals involved in an interaction as observers of their own experiences made up of their own re-presentations and of the occurrences that follow those re-presentations, which they consider as first-order observers (§3). This characterization of first-order observers fits the concept of second-order observers in that it involves awareness of and reflection on regenerations of past experiences that presumably refer in this context to experiences of mathematics teaching. In an ongoing episode of teacher-student interactions, a mathematics teacher can stand back from the interaction and posit situations or tasks "on the fly," a phrase Edith Ackermann (1995) used to describe how hypotheses are formulated in clinical interviews. So, a teacher engages in second-order observation when involved in reflection on regenerations of her own past experiences of teaching students for the purpose of generating situations to engender what is regarded as desirable student activity. Apparently, the authors consider second-order observers as restricted to observing interaction between two other agents that do not include themselves (§3). In the one case where they considered the possibility of an observer observing him or herself, it was as researcher during the data analysis.

« 4 » The authors' account of interaction was basic in their concept of the teacher as negotiator between the learner and the subject content because there was little development of the teacher's model of, in the authors' terminology, the MOS (§4). There are three important distinctions in a constructivist teacher's negotiating between "mathematics" and students' mathematics that were not mentioned by the authors. First, a distinction can be made between a teacher engaging in responsive and intuitive interaction and engaging in analytical interaction (Steffe & Ulrich 2013). In the former case, the teacher is an agent of interaction [and action] who harmonizes herself with the students with whom she is working. She "loses" herself in the interaction and makes no intentional distinctions between her knowledge and the students' knowledge. At this point in teaching, the teacher is not an observer, but an actor somewhat like von Glasersfeld described, where "there is no split between the experiencer and what was being experienced" (Glasersfeld 2009: 257ff). Second, in the case of analytical interaction, the teacher "steps out" of her role in responsive/ intuitive interaction and becomes a first-order observer of her students' language and actions and focuses on analyzing their thinking in ongoing interaction (Steffe & Wiegel 1996). The teacher's interventions are focused on constructing experiential models of the students mathematics with whom she is working. In essence, she becomes the students and attempts to think as they do in analytical interaction (Thompson 1982, 1991; van Manen 1991). These attempts constitute ultimate acts of decentering on the part of the teacher and constitute constructions of meanings of interaction from the students' side of the interaction, which is an essential aspect of constructivist teaching. **How would the authors explain the students' side of an interaction? (Q1)**

« 5 » Finally, in that case where the teacher intends to investigate student learning, the teacher becomes a second-order observer. As a second-order observer, the teacher focuses on creating situations of learning that she might use to engender accommodations in students' ways and means of operating (Steffe 1991b) that explicitly as well as implicitly takes into account the mathematical knowledge of the teacher as well as the mathematics of the students (Steffe & Wiegel 1996). That is, as a second-order observer, the teacher establishes zones of potential construction (Olive 1994: 163; Steffe & Olive 2010: 17ff) for the students prior to interacting with them and, in the context of actual interaction, establishes actual zones of construction that can be used in the teacher's construction of further zones of potential construction.

Students' agency

« 6 » A major goal of the constructivist teacher is to interact with students in such a way that not only allows or permits them to become active agents of action or interaction, but also brings forth and sustains students' more-or-less independent mathematical activity and interactivity, which may transcend the teacher's suggestions or directives (Steffe & Wiegel 1996; Steffe & Tzur 1994). In reading through the text surrounding Grid Algebra, I didn't read protocols or passages that characterized the students in these ways. In contrast, the image portrayed was one where the teacher basically did not relinquish control in that the teacher seemed to be centrally involved in each protocol. **Simply put, had the teacher established the classroom in such a way that he could step out of the classroom for rather extended periods of time with the confidence that the students would remain deeply engaged in goal-directed mathematical activity or interactivity? (Q2)**

Experiential models of students mathematics

« 7 » Following the major goal identified in §6 above, another basic goal of a constructivist teacher is to construct experiential models of students' mathematics and to use those models in the construction of a mathematics for children. There is a major distinction between anecdotal accounts of mathematics of students' and what are considered experiential models. For example, in §26 the authors interpreted the tone of Jordan's comment as indicating that he did not consider negative integers to be multiples of two, like positive integers. Although I find the inference suggestive, extensive investigation in the context of teaching would be needed to explore the depth and breadth of how Jordan might operate in the context of constructing "negative integers" (Ulrich 2012). I have already commented that the bottom line in establishing experiential models is whether the teacher can think and operate as if they are the students, which implies that the students are intensively engaged in mathematical activity. Strictly speaking, a living, experiential model consists of one or more students' characteristic ways and means of operating in a consensual domain.[2] **So, how is a reader to interpret the anecdotal accounts of students' language and actions in the protocols? Did the teacher take them to establish consensual domains among the students? If so, was Dan's comment "The one-times table?" (§29) imputed to the other students in the class? (Q3).**

2| I appreciate that the author's may have had no intention to work with Jordan on negative integers.

« 8 » My use of the phrase "mathematics of students" is somewhat different from the way that the authors use it. I use "students' mathematics" to refer to the general assumption that students indeed have constructed ways of operating that could be thought of as mathematical simply because they are human beings not unlike myself. My use of "mathematics of students" is in reference to students' characteristic ways of operating that I have been able to bring forth in students as they engage in more-or-less independent mathematical activity and interactivity. In the authors' usage, the phrase "mathematics of the students" (MOS) refers to the uninterpreted student. The authors speak of models of the MOS but they do not use a phrase to refer to teachers' knowledge of the MOS, except for "second-order experiential models" (§3). By more specifically using "mathematics of students" to refer to second-order knowledge of students' mathematics, not only does it highlight reference to an experiential model a teacher may have constructed of one or more students, it also highlights the more-or-less independent mathematical activity or interactivity that serves in the construction of the experiential model.

"Mathematics curriculum" and mathematics of students

« 9 » Along with the authors, I separate the teacher's knowledge into two parts, the part that she would not attribute to her students and the part that she has learned by means of interacting with students; that is, in my terminology, the mathematics of students. For clarity, the teacher's knowledge of the curriculum can be considered teacher's knowledge that is complementary to (but perhaps overlapping) her conception of students' mathematics (Steffe 2007). Viewed in this way, that part of "the curriculum" that does not overlap her mathematics of students is just as inaccessible to the students as students' mathematics is to the teacher. Both are examples of first-order knowledge, which are models that subjects construct to organize, comprehend, and control their own experience (Steffe et al. 1983: xvi). The experiential models the teacher constructs to account for what students say and do constitute second-order knowledge (ibid). Likewise, the mathematics the students construct of the teacher's mathematics are the mathematics of the teacher from the students' perspective. So, the problem in mathematics education that the authors address, which is the disconnection between the mathematics curriculum and the MOS, can be recast as the disconnection between the teachers' knowledge of the curriculum and the teachers' knowledge of students' mathematics, which I have referred to as mathematics of students. Recasting the problem in this way conceives of the teacher as an intelligent and adaptive teacher capable of engaging in ethical teaching, and it is compatible with the authors' claim that they are not presenting a curriculum-versus-learner construct.

> "Rather, we propose that just as 'two points define a straight line' (Dewey 1902: 16) so does the 'mathematics' of the teacher and that of the student (MOS) define the course of teaching." (§4)

However, I do believe that the authors would agree with me that Dewey's metaphor is misleading in that the relationship between the students' construction of elements of the teacher's knowledge of the curriculum and the teacher's construction of students' mathematics does not proceed along a linear path.

Final comments

« 10 » The authors opened their article with the question concerning whether there is a kind of teaching called "constructivist teaching." The claim that RC is a theory of learning but not teaching (Simon 1994; Engström 2014) is, in my opinion, a misconstrual because RC is a general model of knowing (Glasersfeld 1995) that has been used in family therapy and other fields. It does contain the constructs assimilation, accommodation, and equilibrium, and these constructs can be used in constructing models of specific aspects of mathematical learning. But the constructs can be used in establishing intersubjectivity among the participants in conversation as well (Thompson 2013). Consequently, we should not attempt to simply *apply* RC to mathematics learning or to mathematics teaching in an attempt to improve either. Rather, mathematics educators must engage in the hard work of constructing models of mathematics teaching that are similar to the available explanations of students' mathematics using the tools of the general model of knowing that are grounded in the experience of teaching students. It is in this context that I agree with the authors that it is the teacher who must negotiate between the teacher's knowledge of the curriculum and the teacher's knowledge of the mathematics of students. In fact, I regard the disconnection between these two aspects of teachers' knowledge as the most fundamental problem of mathematics education today and it is where mathematics education is failing worldwide (Steffe 1996).

Leslie P. Steffe earned his PhD in mathematics education in 1966 from the University of Wisconsin working under the direction of Henry Van Engen. He joined the faculty of mathematics education at the University of Georgia in 1967, where he was appointed Distinguished Research Professor in 1985. Along with Ernst von Glasersfeld and John Richards, a philosopher of mathematics, he established the research program, Interdisciplinary Research on Number (IRON), circa 1976, which continues on through the present time. The book, *Children's Fractional Knowledge* (2010), is one of the latest publications of this research program.

Received: 26 September 2016
Accepted: 29 September 2016