Negotiating Between Learner and Mathematics

A Conceptual Framework to Analyze Teacher Sensitivity Toward Constructivism in a Mathematics Classroom

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> Context • Constructivist teachers who find themselves working within an educational system that adopts a realist epistemology, may find themselves at odds with their own beliefs when they catch themselves paying closer attention to the knowledge authorities intend them to teach rather than the knowledge being constructed by their learners. **> Method** • In the preliminary analysis of the mathematical learning of six low-performing Year 7 boys in a Maltese secondary school, whom one of us taught during the scholastic year 2014–15, we constructed a conceptual framework which would help us analyze the extent to which he managed to be sensitive to constructivism in a typical classroom setting. We describe the development of the framework M-N-L (Mathematics-Negotiation-Learner) as a viable analytical tool to search for significant moments in the lessons in which the teacher appeared to engage in what we define as "constructivist teaching" (CT) during mathematics lessons. The development of M-N-L is part of a research program investigating the way low-performing students make mathematical sense of new notation with the help of the software Grid Algebra. **> Results** • M-N-L was found to be an effective instrument which helped to determine the extent to which the teacher was sensitive to his own constructivist beliefs while trying to negotiate a balance between the mathematical concepts he was expected to teach and the conceptual constructions of his students. **> Implications** • One major implication is that it is indeed possible for mathematics teachers to be sensitive to the individual constructions of their learners without losing sight of the concepts that society, represented by curriculum planners, deems necessary for students to learn. The other is that researchers in the field of education may find M-N-L a helpful tool to analyze CT during typical didactical situations established in classroom settings. **> Constructivist content** • Radical constructivism as espoused by Ernst von Glasersfeld and as interpreted in the context of mathematics education by Leslie Steffe has played a major role in devising M-N-L. The work of John Dewey, Martin Simon, Heinz Steinbring, and David Kolb inspired aspects of the framework as well. **> Key Words** • **Mathematics teaching, mathematics learning, constructivist teaching, constructivist framework.**

Whoever teaches learns in the act of teaching, and whoever learns teaches in the act of learning.
Paolo Freire (1998: 31)

Introduction

« 1 » There are at least four arguments researchers use to contest the existence of a kind of teaching which may be called "constructivist teaching" (CT):

- Constructivism is a theory of learning, not of teaching (e.g., Simon 1994; Engström 2014);
- Irrespective of the teaching approach, learners will come to know by actively constructing mental constructs for themselves (Simon 1994);
- CT is mistakenly equated with progressive modes of teaching (e.g., Engström 2014);
- A constructivist belief does not translate itself into a particular teaching method (Simon 1995).

« 2 » We agree with all these arguments but we still claim that the notion of CT is viable if it is attributed to a teacher's *sensitivity* to learners' active and subjective construction of knowledge during the teaching-and-learning process. As part of a larger research project, we have developed an analytic framework (M-N-L) which may be used to investigate CT in a typical lesson within a school setting.

Key terms

« 3 » The M-N-L framework and its application to analyze CT makes use of a number of key terms which need to be defined at the outset. These definitions are our own understandings of the meanings of these terms:

http://constructivist.info/12/1/059.borg

"Mathematics" or **"subject matter"** is the consensual domain (in the sense of Maturana & Varela 1980; Glasersfeld 1991a) of mathematical concepts and skills existing among persons who have come to internalize them through personal experiences, and who are capable of utilizing them to make sense of further "similar" experiences and to use them as a scaffold to deepen their understanding of those concepts and skills and to learn others. Such "mathematics" is agreed upon and communicated within mathematical communities (such as in a mathematics classroom) by building a consensus in certain aspects of the subjective realities of the persons belonging to those communities.

Teacher is the person who, besides being knowledgeable in subject matter, is also knowledgeable in the art of helping other persons (students) to construct meanings, and establish a consensual domain with them in such a way that they can communicate each other's knowledge through interaction.

"Interaction" is the act of re-presenting (Glasersfeld 1991c) internalized concepts through various external expressions (such as utterances, body movements, diagrams, and symbols) which are interpreted by other persons in relation to their experiential world. The persons taking part in an interaction, such as the teacher and the students, are "first-order observers" of their own experiences made up of their own re-presentations and of the occurrences that follow those re-presentations.

"Learner" may be taken to be synonymous with "student," but while a "student" is such because he or she attends a lesson (usually in a school setting), a "learner" is such because he or she *learns* something from that lesson. Only the learners themselves have "control" over and first-order knowledge of their own learning and so, while the M-N-L framework assigns the status of "learner" to each and every student, actions done by the teacher are discussed in relation to "students." This is a subtle, but important change of terms to indicate that the teacher strives for learning but the act of learning is up to the student.

"Second-order observer" – a term first used by Humberto Maturana (1978) – is a person who "observes the occasions of observing of the first-order observer" (Steffe & Wiegel 1996: 483). As observers of the interactions between teacher and students, we regard ourselves as second-order observers, where we can only hypothesize about the mental states of the teacher and the students with reference to our (mathematical) consensual domain. Being the teacher-researcher, one of us took on the roles of both first- and second-order observer, the former as teacher during the lessons and the latter as researcher during the data analysis. Since much of what is reported here is presented from his perspective as a teacher-researcher, the analysis is mostly an expression of self-reflexivity.

"Mathematics of the students" (MOS) is the "mathematics" inside students' minds and hence no one, except the learners themselves, can have access to it. A teacher may form second-order experiential models (Steffe et al. 1983) of MOS by observing students' external actions, compare them to other "similar" actions, including his or her own, and hypothesize about the mental state of the students. By engaging students in mathematical discussion and activities, the teacher establishes viable models of students' syntactical and lexical mathematical meanings (Thompson 2013).

"Mathematics for the students" (MFS) is the mathematics the teacher intends to teach to a particular group of students. It includes the teacher's models of "mathematics" of other students "like" the current students he or she has gained through teaching experiences and also the teacher's "mathematics" he or she would not attribute to the current students. The teacher's formation of MFS is the settlement of the continual perturbations created by tensions arising from these two kinds of "mathematics" and based on his or her interpretation of MOS.

"Curriculum" – as used in this report – is the program of teaching and learning a body of subject content determined by curriculum planners in educational authority positions and made obligatory in schools. In educational contexts that adopt a realist epistemology, such as that of the participants of this study, the curriculum is handed down from authorities as a body of *a priori* knowledge. RC teachers, however, believe that no such thing exists and that the curricular topics they are required to teach are actually knowledge they (the teachers) have construed for themselves. In particular, "teachers' mathematics" is the "mathematics" (as defined above) teachers have constructed from their experiences as learners, combined and enriched with MOS and MFS they continually assimilate from their experiences as teachers.

Description of the M-N-L framework

« 4 » The Mathematics-Negotiation-Learner (M-N-L) framework proposes a theory about what CT might mean by focusing on the role of the teacher in the educative process and portrays him or her as a *negotiator* between the learner and the subject content (hence the dashes in M-N-L). This does not mean we are presenting a curriculum-versus-learner construct. Rather, we propose that just as "two points define a straight line" (Dewey 1902: 16) so does the "mathematics" of the teacher and that of the student (MOS) define the course of teaching. We view the teacher as a *guide* who walks with students in their journey towards the formation of a consensual domain that contains the teacher's "mathematics," which forms part of the curriculum and is designated by the teacher as "mathematics for the (current) students" (MFS), but which also contains MOS. An important derivative of this process is that the teacher's "mathematics" is deepened by his or her reflections on MOS. The constructivist teacher learns in the process of teaching and thus, the consensual domain between teacher and learner (and, possibly, other learners in the classroom) is not only a construction of the learner but also of the teacher. Figure 1 illustrates the dynamics of such a negotiation, where "negotiation" is demonstrated by a left-right arrow connecting "mathematics" and learner.

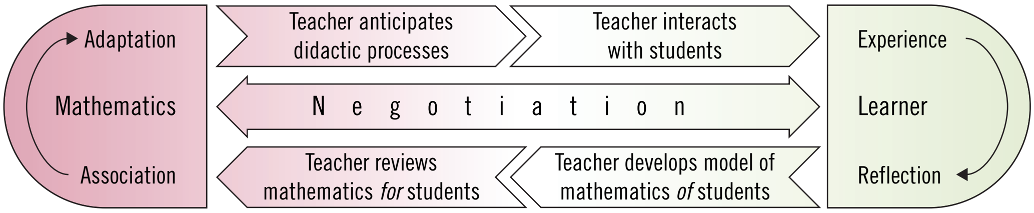

Figure 1 • The Mathematics-Negotiation-Learner framework.

« 5 » We use the metaphor of a two-way road to explain the process of negotiation, with two stages going one way and two stages going the other way. The following is a description of the stages starting from the upper left-hand arrow going from "mathematics" to learner:

A Building on knowledge of MOS of previous similar students, the teacher anticipates possible didactic processes that can enable the current students to conceptualize and internalize MFS, a "subset" of the teacher's mathematics." Martin Simon (1995) calls this a hypothetical learning trajectory, since the teacher has no way of knowing the learning processes in advance.

B The teacher interacts with the students according to his or her anticipations of the didactic processes. The interaction can take many forms, including:
- Questioning the students to stimulate discussion and to create second-order models of their thinking processes and of the MOS;
- Answering students' questions and elaborating on their statements to encourage students to think more deeply about MOS;
- Introducing and setting goal-oriented activities (through demonstrations and discussions), intended to help students gain access to MFS and guiding the students in those activities;
- Giving feedback to students about the outcome of set activities.

Besides the more progressive goal-oriented activities, our understanding of "interaction" thus includes teacher exposition (plenary approach), which a second-order observer might consider "traditional" and "non-constructivist." What constructivist teachers bring to the table that would distinguish them from non-constructivist ones is that their actions are intended to form second-order experiential models of their students' thinking. This may be observed (by second-order observers) through teacher expressions such as "*Tell me what you think about this*," or "*Ask me something about this*," even in "traditional" plenary teacher expositions.

C The "Learner" section of Figure 1 shows how this interaction helps students *experience* some form of mathematical phenomenon that the teacher encourages them to *reflect* upon ("*Now, why do you think that happens?*"), sometimes with the help of further experimentation on that experience. Students are bound to make abstract conceptualizations which result from this interplay between experience and reflection, a process by which they become "learners." This is reminiscent of David Kolb's (1984) *Experiential Learning* construct but the emphasis here is on what constructivist teachers do with the manifestations of learners' mathematics.

D The first arrow moving from learner to "mathematics" illustrates the result of the constructivist teacher's first-order observation of students' re-presentations of their knowledge, which they do as a result of reflecting upon their experiences. The teacher uses this observation to create second-order models of MOS.

E Without abandoning the curriculum or his or her "mathematics" the teacher uses these models of MOS to review MFS (Steffe 1991a). "Review" is a rather mild term which we use to denote the teacher's entry into the "mathematics" perturbation that follows.

F The "Mathematics" section of Figure 1 signifies the perturbation that the teacher goes through when faced with the tension created between his or her original MFS and the models of MOS. Sometimes this perturbation is settled easily by simply assimilating MOS with the intended MFS. At other times the teacher needs to adapt his or her MFS in order to cater for MOS (accommodation). The teacher's equilibration (Piaget 1985) of MFS, which forms part of his or her "mathematics," renders teachers learners themselves.

« 6 » The structure of M-N-L may give the impression of a cycle of stages occurring in a neat order throughout the lesson such as Simon's (1995) model of teaching from a constructivist perspective or Heinz Steinbring's (1998) construct of teaching and learning processes. We propose that the stages of the M-N-L framework are only indicative of the order in which they occur and the two ways in which "mathematics" is connected to the learner. In the course of a lesson a teacher might skip a stage or revert to a preceding stage if he or she feels the need to focus on one aspect of the negotiation dynamics.

« 7 » Something which distinguishes M-N-L from both Simon's (1995) and Steinbring's (1998) models is that the teacher-learner interaction does not only help the

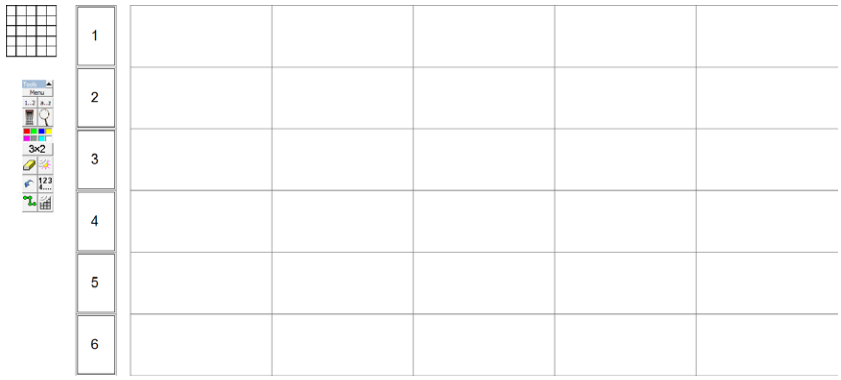

Figure 2 • The GA software blank interface.

teacher form hypothetical learning trajectories (Simon 1995) or inform the teacher about the appropriateness of further learning offers (Steinbring 1998), but also enriches the teacher's mathematical content knowledge. This is where we disagree with Steinbring's (1998) claim that the act of mathematics teaching can be regarded as an autonomous system. While learners, even though they may be better or worse off with the interventions of the teacher, construct mathematical ideas in a relatively autonomous way, teachers can never regard their actions in the classroom as being independent of the learners. What M-N-L proposes is quite the contrary. CT is almost totally dependent on students' feedback. Even if autonomy of teaching *approach* is granted by school authorities, teachers engaging in CT need to allow the learning environment they set up to be affected and, to a certain extent, determined by what they learn from their students.

« 8 » We propose that a teacher engaging in CT will show sensitivity to constructivist notions when he or she is observed to try to negotiate a "road" that connects the "mathematics" of the teacher to that of the learner. All this would not be possible without the teacher's *sensitivity* to students' mathematical understandings (Jaworski 2012). Consequently, we regard the teacher's lack of sensitivity or deliberate disregard of this sensitivity to be a roadblock in one of the two paths between "mathematics" and learner: the teacher either fails to let his or her mathematics be influenced by knowledge about learners' "mathematics" or fails to interact with the learners in a way that encourages active participation. Before demonstrating how second-order observers may use M-N-L to investigate CT in a typical mathematics lesson, we will now describe the classroom context and participants of the present research study.

The research context and participants

« 9 » The group of participants consisted of six Year 7 boys (pseudonyms: Dwayne, Tony, Omar, Jordan, and Joseph, being 11-year-olds and Dan, a 12-year-old) attending a single-sex secondary school where the teacher-researcher had been working for 18 years as a full-time mathematics teacher.[1] In the subjects of Mathematics, Maltese (mother language), and English (second language) students in this school were divided into three groups according to their performance in these particular subjects. The participants formed the lowest-performing group in Year 7 mathematics. This performance was measured from their scores in a national benchmark examination at the end of Year 6. In the mathematics examination, the participants' scores ranged between 1 and 3 standard deviations below the national mean.

« 10 » A substantial part of the research project concerned the use of software called Grid Algebra[2] (GA) which is a grid-based computer environment built on the multiplication table grid. GA was used due to its potential to help students conceptualise the meaning signified by standard mathematical notation, something which we regard as crucial in students' introduction to formal algebra. Although the focus here is not GA, an explanation of some of its features is necessary to understand the protocols of lesson episodes included later on. Figures 2–4 show GA computer interfaces. The first, Figure 2, shows a blank interface with cells representing multiples of 1–6. An integer may be chosen from a number menu and inserted anywhere on the grid, as long as it is a multiple of the number showing at the beginning of a row.

« 11 » Once a number is placed inside a cell it determines what numbers could be entered in the other cells. As shown in Figure 3, entering "5" in the second column of the *first* row means that the number that could be entered in the second column of the *third* row has to be "15" (5 times 3) and the one in the second column of the *sixth* row has to be "30" (5 times 6). An important feature of GA is that the cells (containing numbers or expressions) may be moved around the grid. When a cell is dragged onto another, the number or expression in the original position changes according to the mathematical operation associated with that movement:

- right movement corresponds to addition;
- left movement corresponds to subtraction;
- downward movement corresponds to multiplication;
- upward movement corresponds to division.

« 12 » Figure 4 shows the result of such movements. The new position of a cell does not show the single numeric "answer" of the operation, but the "answer" is shown by GA as an expression showing standard mathematical notation. So, for example, dragging a cell containing "15" in the third row to the

1| The first author was the teacher-researcher in this study up till the scholastic year 2014–2015. For the sake of readability, he will be mostly referred to as "the teacher."

2| This software was developed by Dave Hewitt. Readers are invited to view short demonstrations of how it works by visiting https://www.youtube.com/watch?v=HmVjprJWInM and the subsequent links (Grid Algebra 1–4).

sixth row (corresponding to a multiplication by 2) will result in the expression 2(15) or 15 × 2 (depending upon a chosen setting). Successive cell movements produce more and more complex expressions. There is also the possibility to assign a letter to a cell. As shown later on, a letter may represent a variable (a general number) or an unknown (a specific number) depending on the situation. Moving a cell with a letter results in expressions similar to those obtained when moving a cell with a number, the only difference being that the resulting expressions would be formal-algebraic rather than just numeric. Moreover, when a cell is the destination of more than one journey, it stands for more than one expression. When this happens the software allows the user to see the equality of the two expressions, such as 2(5 + 1) = 10 + 2. This feature was used to help students understand that (a) an "answer" is not necessarily a single number (it can be an expression) and (b) the number on the right of the equals sign is not necessarily the result of the calculation on the left (in the above equality, 10 is not the result of 2(5 + 1)).

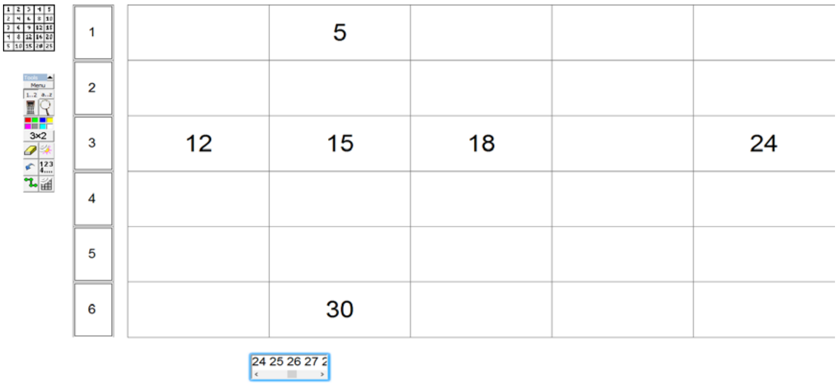

Figure 3 • GA cells with numbers inserted in respective multiplication tables.

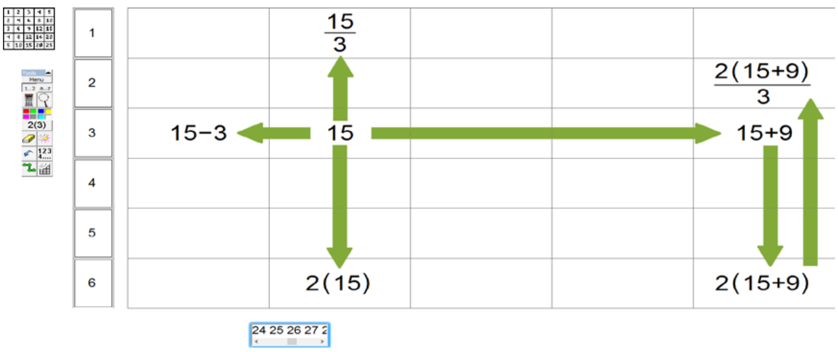

Figure 4 • Movements of GA cells to new positions to obtain new expressions.

Data collection and analysis

« 13 » The research extended over the scholastic year 2014–2015, where students were given twenty double lessons (80 minutes each) using GA. These lessons formed part of the curricular lessons for that particular scholastic year. The rest of the lessons did not form part of the research data, although the teacher-researcher (acting only as a teacher in these other lessons) may have used observations from these lessons to serve him for building second-order models of MOS in the research lessons.

« 14 » In the research lessons data was collected through video recordings of the lessons, computer-screen recordings of students' work, students' pen-and-paper work, and video-recorded interviews (five per student throughout the year).[3] The first part of a typical lesson consisted of a class discussion about the topic of the lesson which included plenary demonstrations of GA activities on an interactive whiteboard by the teacher which he usually made with the participation of the students themselves. The second part consisted of students working in pairs on their computers where the teacher went around and assisted the students where necessary.

« 15 » Without assuming any particular teaching method, the teacher-researcher wanted to answer this question: How can GA be used as a tool for constructivist teaching? This question was divided into two more specific subsidiary questions:

- What are the significant moments in the lessons where the teacher can be observed trying to teach in a way that is sympathetic to constructivist notions of learning?
- What are the significant moments in the lessons where the teacher can be observed to be less sympathetic to constructivist notions of learning?

These questions were answered through the formation of categories and themes with reference to the M-N-L framework.

« 16 » An important aspect of the whole research project was that the lessons that provided the data were not a teaching experiment in the classical sense (such as Steffe 1991a). The teacher-researcher did not try out something new (he had been using GA for some years with other groups), the lessons formed part of the yearly scholastic scheme of work for that particular group, and none of the class was singled out or excluded. The research was principally an exercise of reflection about the teacher's teaching and about the students' learning with the help of insights from relevant literature. The lessons themselves, the teaching interventions, the students' actions, and the overall dynamics of the teaching episodes served as a stimulus for the generation of data, pertaining to the issue of CT, for this reflection and analysis.

3| Only preliminary analysis of the lesson videos and the computer-screen-capture videos will be used for this report.

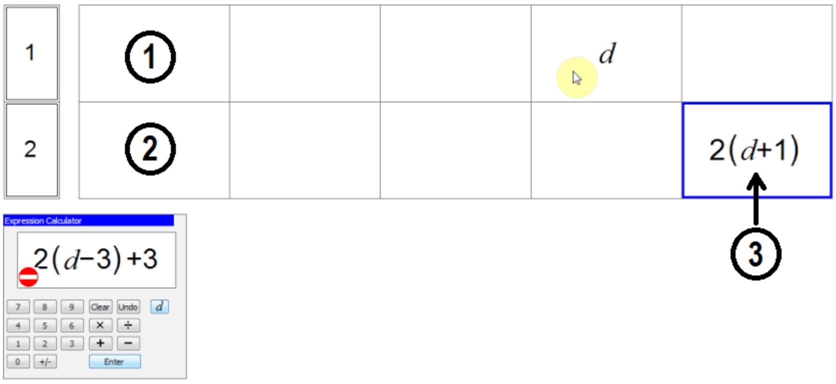

Figure 5 • Screen capture of Omar's work on GA.

Evidence from preliminary data analysis

« 17 » Preliminary data analysis of the lessons led to the emergence of two central themes in relation to CT:

A The shifts of focus during CT in order to negotiate relationships between the learners and "mathematics."

B The teacher's loss of sensitivity towards constructivist notions when one of the negotiation paths on the "road" between learners and "mathematics" was blocked.

A. Teacher as negotiator between learner and mathematics

« 18 » This theme was developed from four categories which correspond to shifts in the teacher's focus as he was trying to establish one of the two paths on the road that connected his "mathematics" with that of the learners:

I from the learner (experience and reflection) to the negotiation process (teacher's formation of models of MOS and review of MFS);

II from the negotiation process (as in I) to "mathematics" (association / adaptation of mathematical content intended to be taught);

III from "mathematics" (as in II) to the negotiation process (teacher's anticipation of possible didactic process and interaction with students);

IV from the negotiation process (as in III) to the learner (as in I).

« 19 » The order of these shifts of focus are according to the M-N-L framework but there is no starting or finishing stage. In a lesson, the teacher may start with IV and move on to I, II, and III, or start with III, IV, and so on. Moreover, the number of times the teacher shifts his or her focus in a lesson is indefinite. In the following four episodes, we attempt to demonstrate moments in different lessons when the teacher seemed to make such shifts of focus in order to engage in CT. This is done with the help of protocols of lesson conversations with reference to preceding or subsequent moments in the lesson. Protocols are accompanied by diagrams showing the GA-screen interface at the moment of the protocol. Some of the episodes are goal-oriented moments. Others are situations where the teacher and the students were engaged in whole-class discussion.

Shift I: Learner-to-Negotiation

« 20 » In the *Learner-to-Negotiation* shift, the teacher observes the way that students are dealing with a particular experience and the way they express their reflections upon that experience, and uses this observation to form second-order models of MOS, which he or she may later use to review and possibly modify MFS. In one of the lessons the students were involved in a GA task where a letter was given in a random place on the grid and a random empty cell was highlighted. Their goal was to use an "expression calculator" to formulate a mathematical expression involving that letter that could fit in the highlighted cell. The students were expected to construct three different algebraic expressions for that particular cell before GA presented them with a new problem. Figure 5 shows a screen-capture video still of the computer being shared by Omar and Jordan, who were taking turns in trying out the problems.

« 21 » The "mathematics" required to solve this problem was primarily the knowledge of the operations that correspond to particular, ordered, horizontal or vertical movements along the multiplication grid. This activity was intended to familiarize students with the standard notational convention that designates a particular order of operations for particular mathematical expression. For example, $2(d+1)$ corresponds to the addition of 1 to "*d*" followed by a multiplication by 2. Traditionally, students have been offered mnemonics such as PEMDAS to remember the convention of the order of operations signified by a mathematical expression (in this case, parenthesis needs to be worked out before multiplication). GA contains inbuilt tasks where students are expected to construct an expression that corresponds to a particular order of operations (movements along the grid) such as the task shown above. It also contains inverse tasks, where students are expected to show the order of operations for a given expression.

« 22 » At this particular moment it was Omar's turn to try a problem and he had already got one correct answer out of the expected three. Analysis of the video shows that he got to the expression $2(d+1)$ by imagining the cell containing "*d*" to move one cell to the right (plus 1) and *then* move from the one-times table to the two-times table (times 2). The circled numbers (1, 2, 3) are superimposed on the video still to show where Omar was pointing during Protocol 1. Just before the episode, Omar had tried out the second algebraic expression for the highlighted cell but got a "no-entry" signal (Figure 5), meaning that his expression was wrong and was not being accepted in the cell. At that moment the teacher was moving around offering his assistance to the students and he turned his attention to Omar, who seemed to be frustrated.

Protocol 1: Omar's mistake

PB: [Notices that Omar got frustrated] What's the matter there, mate? [...]
Omar: I, actually, I did the "*d*," I will go here...
PB: Show me with the mouse [...] you said, "*d*" will go there [position 1] good.
Omar: I'll move over here [position 1], go down here [position 2],...
PB: Good.
Omar: And then I go over here [position 3]
PB: Good. But... Tell me how much plus you do from here [position 2].

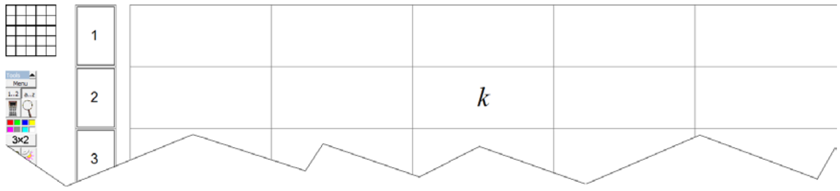

Figure 6 • Reproduction of interactive whiteboard showing a letter in the grid.

« 23 » At this moment the teacher shifted his focus to establish a model of Omar's MOS. Further discussion suggests that the teacher noticed that from position 2 to position 3 Omar was adding in steps of 1, rather than 2, as expected. Omar himself realized he was making another mistake, that he was not including the designated cell in his counting (he had written +3 instead of +4). The ensuing dialogue suggests that the teacher was comparing Omar's MOS with the teacher's own "mathematics" (that moving in the two-times table meant adding twos) and helped Omar to realize that in order to move four cells to the right he had to finish his expression with +8 (not +4).

Shift II: Negotiation-to-Mathematics

« 24 » This transition refers to the several occasions where constructivist teachers act upon their model of MOS to review their MFS by assimilating / accommodating it in their "mathematics" schemas. The following episode is not taken from a goal-oriented section of a lesson but from a class discussion where the teacher was checking out whether students were prepared to learn about the concept of a variable and about the distinction between a variable and an unknown.

« 25 » As previously explained (Figure 3), when a number is inserted in the GA grid it predetermines the numbers that could be placed in the other cells. Placing a letter in such a grid would mean that the letter represents a constant number, an *unknown*. For example, placing "n" in the second-row, second-column cell in the grid of Figure 3 would mean that "n" represents an unknown (n = 10). Prior to the episode presented here, at the beginning of the lesson, the students had been working on possible values that a letter in a GA cell could signify if other numbers were already on the grid. The teacher wanted to see what they thought if a letter was placed in an empty grid. On the interactive whiteboard, he inserted "k" inside a random cell, as shown in Figure 6, and asked the students what they thought about it.

Protocol 2: The letter symbolizing a variable

PB: What number is *k* symbolizing? [Joseph and Dwayne put up their hands eagerly]
Dwayne: [Shaking his finger] Sir. Sir. I know. I know.
PB: Be careful! Be careful and think a bit before replying.
Tony: [Without raising his hands] Any number.
(At the same time that Tony spoke PB was giving permission to Dwayne to speak)
PB: [Gesturing towards Dwayne] Come, let's see.
Dwayne: Any number [here he is joined by Joseph and they talk together] that lies in the two-times table. [Looking at each other] Jinx!...

« 26 » The ensuing discussion suggests the teacher was trying to develop a model of MOS by asking the students to give examples of what "k" might symbolize. When it was established that the students agreed that "k" could stand for any multiple of 2, the teacher decided to introduce the term "variable" and tried to demonstrate the notion of a variable by scrolling along the numbers menu provided by GA. When negative numbers started appearing, Jordan said, "*Eh! There would be the negative numbers!*" The tone in Jordan's exclamation seemed to be interpreted by the teacher as meaning that Jordan (and possibly other students) did not consider negative integers to be multiples of two like positive integers (which is what multiplication tables show). Comparing to his "mathematics" that if a variable is a multiple of 2, then that variable could also take on negative values, the teacher set out to ask students what would happen if one skipped backwards on the two-times table and continued skipping beyond 2 and 0. This developed into a discussion about negative multiples of 2, and it was established that the two-times table actually stretched indefinitely on both sides of the number line.

« 27 » The *Negotiation-to-Mathematics* shift was done twice here by the teacher. The first shift occurred from developing a model of the students' concept of variable to introducing the term "variable" and discussing it with them in terms of multiples of 2. As soon as the teacher started offering a learning experience by scrolling the numbers menu, Jordan made a comment that the teacher seems to have interpreted as a combination of surprise and realization. The teacher assimilated this model of MOS to his original "mathematics" of a variable and set out to discuss why varying multiples of 2 were not limited to positive integers.

Shift III: Mathematics-to-Negotiation

« 28 » Every time the teacher-researcher introduced something new about GA, his approach was to project some image from GA on the interactive whiteboard and start a discussion about what the students were observing on the screen. This involved at least two anticipations of the didactic process. The first was his anticipation of possible didactic processes through which he might

Figure 7 • GA's first row of cells with numbers.

set out to present MFS to the students. Some decisions about MFS were planned before the onset of the lesson according to models the teacher had previously built of the current students' "mathematics" and about experiential models of MOS of past students who were "similar" in age, grade, and performance level to the current students (the settlement of the "mathematics" perturbation). Some decisions about MFS had to be done in the course of a lesson. The other anticipation was about the possible, viable interactions that could enable those didactic processes.

« 29 » The following episode is taken from the lesson in which the students experienced the GA software for the very first time. The teacher started off by showing them just one row of cells with numbers (Figure 7). The "mathematics" the teacher intended to discuss with the students was that the first row of numbers (1, 2, 3, 4, 5) formed, among other things, the first row of the multiplication tables (the one-times table). The teacher's goal of the first lesson was to familiarise the students with GA and he decided the best way to do this was to present the GA grid as a snapshot of the multiplication table grid. Past experience with "similar" students showed that starting with many rows made it hard for students to focus on individual rows and so the decision for MFS at the start of the first lesson was to show only the first row, interact with them (in this case through demonstration and discussion) and discuss what they thought about those numbers by asking questions that deliberately helped students fall back on their experience of such a number sequence. Protocol 3 follows right after the teacher ran the GA software and set it with the first row of cells filled with numbers, as reproduced in Figure 7.

Protocol 3: Making a learning offer

PB: First of all, are you noticing what these are?

Dwayne: Yes.

PB: What are those that you have in the whole row?

Dwayne: It is a row with numbers.

PB: Okay, tell me where you've seen it [the row] before? Have you ever seen it somewhere else? Not just a row of numbers. Mention where you've seen it before.

Omar: In a grid.

PB: In a grid of what?

Omar: Of the numbers. When you have all the numbers.

PB: When you have all the numbers. The number grid.

Omar: When you, when you learn to say the tables and the numbers.

[...]

PB: Stay on what Omar said... He said, "I have seen these before in a number grid, when learning the tables." So that [pointing to the first row], don't you think that it is something...? [Dan raised his hand, PB nods towards him].

Dan: The one-times table.

« 30 » Probing questions (anticipation and interaction) usually led to a reflection beyond what was immediately apparent. It is interesting to note how these students built on one another's statements, each time taking the discussion to a higher mathematical level. For Dwayne it was just a row with numbers, further probing helped Omar "realize" that these numbers were found in the number grid he had used to learn the multiplication tables, and further interaction led Dan to state that what they were seeing was the one-times table. The students' "mathematics" progressed from "a row of numbers," to a set taken from the table of multiples, to a particular multiple set – the multiples of 1. The teacher built on this model of MOS to introduce, in this row, large numbers that are not usually seen in multiplication tables (but are still multiples of 1). The lesson then progressed with a discussion about the second row and the relation between the numbers in the first row and those in the second row.

Shift IV: Negotiation-to-Learner

« 31 » During this transition, the teacher's focus turned to what the learners were *experiencing* and *reflecting* as a result of his interactions. This shift was at once interesting and demanding for the teacher because the diverse reflections of individual students could, at times, prove to be overwhelming. Inevitably, this made the teacher negotiate from the students back to his "mathematics" to strengthen or renew the pathways on the "road" between his "mathematics" and that of his students.

« 32 » In the following episode, taken from a lesson where students were quite familiar with GA, the teacher used one aspect of GA to help students focus on the meaning of the equals sign. The "mathematics" that the teacher was aiming to teach here was that besides following a calculation and preceding an answer, the equals sign also showed equality, which could resemble a balance between the left- and the right-hand sides of an equation. The teacher was aware that this was still a restricted concept of the equals sign[4] but he intended to use it to help students start expanding their meaning of equality. At this point in the discussion, the teacher was using the metaphor of the see-saw to help students consider that balance could only be kept if each side of the see-saw had *equal* weight. Dan raised his hand to speak…

Protocol 4: Equality of weights on a barbell

Dan: Like those [meaning weightlifters], don't they lift that iron bar like that [stands up and takes a weightlifter pose with his hands up]?

PB: [Nodding] The iron bar. Well done!

Dan: If on this side [gestures towards his left hand] you have much more, then this side [left] will topple [makes a toppling motion – Figure 8] and he won't be able to keep the balance.

PB: Well done. Well done. We have another example of "equals"...

« 33 » The teacher proceeded to elaborate the discussion about the equals sign using Dan's weightlifting analogy. We make

4| There are at least eight different mathematical uses of the equals sign (see, for example, Usiskin 1988 and Jones & Pratt 2012).

two observations here. The first is that Dan seemed to find a *viable* analogy between "equals" and the balance of weights on a weightlifter's bar, an example of how a mathematical concept may be re-presented in terms of an experiential occurrence. Furthermore, Dan acknowledged the conditions necessary for a balance by showing what would happen if these conditions were *not* met: a toppling of the barbell.

« 34 » The second observation is that the teacher claims he had never used this analogy before and Dan's contribution to the discussion enriched the teacher's "mathematics" because it added to his repertoire of balance metaphors that could be used to help other students become acquainted with the balance notion of the equals sign. His use of the see-saw metaphor (right before the episode) was an attempt to use an example from the probable experiential realities of the current students since experience of past students had taught him that the long-standing metaphor of a balance scales had become almost obsolete in the daily experiences of today's students due to the current rarity of such an instrument.

« 35 » In the following section we show how M-N-L was found to be helpful in identifying moments in the lesson where the teacher appeared to lose his sensitivity to constructivist notions during the lessons.

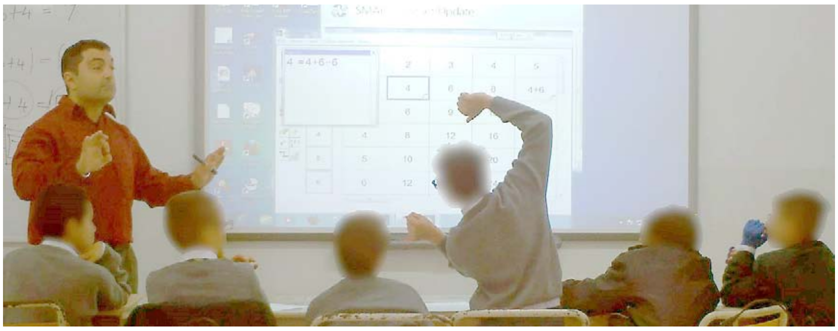

Figure 8 • Dan's weightlifting analogy.

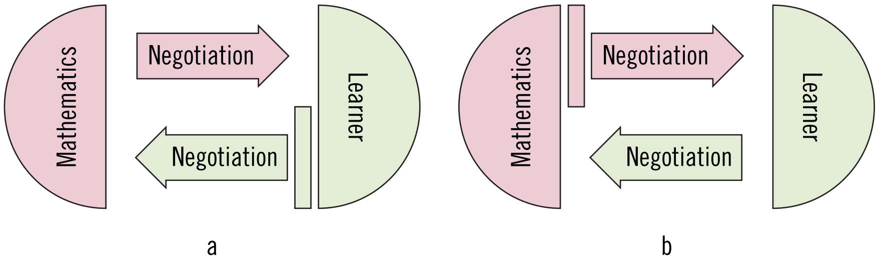

Figure 9 • Roadblocks in the mathematics-learner negotiation paths.

B. Teacher as a barrier to negotiation

« 36 » Building second-order models of MOS requires *purposeful* observation of the interaction of the students with the learning experience. Such an observation is only possible when the teacher is sensitive to constructivist notions of learning. Data revealed that even a self-claimed RC teacher could create barriers to the negotiation of teacher and learner knowledge when he lost or ignored this sensitivity. Figure 9 summarizes how the M-N-L framework helped to identify two ways in which the teacher seemed to fail to engage in CT by barring one of the two negotiation paths on the mathematics-learner "road":

A from the learner side, or
B from the mathematics side, where the black line symbolizes the "roadblock" in that path.

« 37 » There were instances in some lessons where the teacher was observed not to elaborate on a learner's response or deliberately stopping the discussion. We regard this as a barrier in the negotiation road from learner to mathematics (Figure 9a), where the teacher seemed to inhibit learners from teaching him something about their "mathematics." The following protocol is taken from an episode in the first lesson, where the teacher was showing the students what numbers could be allowed to exist in the first row of the GA grid, by scrolling the numbers menu.

Protocol 5: Stopping a discussion

PB: …Over there we are going a bit further away as well. [PB scrolls backwards a bit and showing 0 and some negative numbers and scrolling quickly back to 1]. I am interested from one.
Joseph: Or from minus one.
PB: From one. But I'm interested from one. Do not take notice that we did, that there was zero as well. When the time comes we will do that as well. [PB continues with what he was originally doing].

« 38 » The teacher said that, at that moment, he was aware of missing the opportunity to learn about Joseph's mathematics, and possibly to review his own intended MFS. At that particular moment he refused to "try and build up a model of the particular student's own thinking" (Glasersfeld 1991b: 178) and therefore, from an RC point of view, he could not attempt to strengthen or modify Joseph's conceptual structures. The teacher also missed the opportunity to foster the student's motivation (Steffe 1991a) to talk and possibly learn about zero and negative numbers. This momentary roadblock between learner and negotiation originated from the nature of the comment of the learner (Joseph) which, at the moment, the teacher felt as *threatening* the direction he intended for that part of the lesson: a focus on the first few natural numbers and to acknowledge them as the one-times table. The teacher was aware of this failure on his part and at a later stage in the lesson, when the issue of negative numbers cropped up again, he dedicated some time to talk about the need for negative numbers (such as underground car parks or levels shown inside an elevator).

« 39 » The M-N-L framework helped to identify a similar but distinct type of barrier in the mathematics-learner link. It originated when the teacher was so focused on his "mathematics" that he forgot about the learner altogether. We regard this as a block from the "mathematics" side of the negotiation road (Figure 9b) because the obstruction was caused by the teacher's missing the opportunity to interact with a learner due to a "*single-mindedness*" on the teacher's "mathematics," rather than perceiving a "threat" in the learner's response (as in the first barrier). The following episode shows such an instance in one of the lessons.

« 40 » The protocol is taken from a lesson where the teacher was discussing with the learners what happens when a number in a cell is dragged downwards to another row. At this moment, two students Dwayne and Tony were assisting the teacher in demonstrating what happens when a number in the first row is dragged to the third row (multiplied by 3) and the resulting expression is again dragged to the sixth row (multiplied by 2).

Protocol 6: Too much focus on "mathematics"

PB: So. Let's go. [pointing to the cell with a "10" and then to the cell in the same column, row 3] If I lower this here what will it become?

Dwayne: Te…, it becomes…ten times three.

PB: [PB moves the "10" onto the cell in row 3 and GA transforms it into 10x3]. Good. And if the "10x3," you put it here [pointing first on the cell with "10x3" and then to the cell in the same column, row 6], what does it become? [pointing to the numbers on the left] Look from here, look at the side [Jordan raised his finger and wiggled it]… Three times how much so that… Look from here.

Dwayne: Ten times two,… times…one.

PB: So, "10 × 3" is going to stay there.

Dwayne: Aha [agrees]

PB: So "10 × 3" is going to appear for sure.

Dwayne: Aha [agrees].

PB: Now it [making a gesture like grabbing something], something is going to happen to it.

Dwayne: It is going to change.

PB: It is going to change and something will be added to it [meaning to the expression]. But it will stay there, the "10 × 3."

Dwayne: Aha [agrees]. Ten times two…

PB: Ten times two! Why are you saying ten times two? Ten times three is what you have…

« 41 » Hindsight and second-order observation allow us to suggest that Dwayne could have been saying "ten times two" because he was thinking about the multiplication required for a multiple of 3 to become a multiple of 6 ("jumping" from row 3 to row 6). The "correct" expression was $2(10 \times 3)$ which is actually equivalent to "*ten times two* times three." The teacher, however, did exactly what Dewey (1902) warned against: he focused too much on the *content* he intended to teach and forgot about the conceptual needs of the student. The teacher was observed to *insist* on his "mathematics" even by his (apparent) disregard of Jordan's wiggling finger. If Jordan had been allowed to contribute, the teacher could have observed how he interacted with Dwayne and used that observation to create a model of Dwayne's and Jordan's MOS.

Conclusion

« 42 » The few lesson snippets presented in this article were used to demonstrate how the M-N-L framework was found to be viable in the analysis of a computer-aided lesson to identify instances where a teacher was sensitive to constructivist notions of learning and also instances where he failed to be. Without prescribing a particular teaching method, the framework places the teacher as a negotiator between the subject matter and the learner and suggests that the main task of constructivist teachers is to find ways of building roads between their subject content knowledge (in this case mathematics) and that of their students. In order to do so they must

a maintain a sensitivity, flexibility, and openness to learn about the mental constructs and reasoning of their students,
b learn about, review, and possibly adapt the subject matter they intend to teach,
c anticipate and plan for purposeful interactions with the students, and
d provide students with opportunities to experience and reflect upon phenomena presented by the topics of the subject matter.

« 43 » These four elements show that deciding whether one's teaching may be called CT is not a simple dichotomous matter of determining whether the teacher takes or does not take into account learners' mental processes. As we hope to have demonstrated, teachers who strive to establish CT in their daily lessons also need to take into consideration the subject content that they are *expected* to help their students gain access to and associate it with the students' experiential realities through ongoing interactions with the students themselves. The M-N-L framework shows the complexity that such a task entails in the context of mathematics teaching and learning. In order to engage in CT, teachers have to be flexible and responsive enough to shift their focus continuously during the lesson such that none of the above four elements is neglected.

« 44 » Although the M-N-L framework has been devised from and for the analysis of a mathematics lesson, we believe it can be found viable by teachers and educational researchers in other disciplines, especially in situations where the curriculum is developed by persons other than the class teacher and where it specifies a well-defined set of topics to be taught for specific levels in schools. For instance, a similar *S*-N-L framework for the investigation of CT in science lessons may help teachers reflect on their sensitivity towards their students' construction of ideas about physical phenomena and the functional relationships and regularities (Peschl 2001) among those observed phenomena. Physical phenomena and the relationships between them may be perceived by science teachers as viable models to help students learn how scientists predict and control certain aspects of our experiential realities. A similar parallel framework, *L*-N-L, may be found viable to analyze CT in the teaching of languages, where teachers learn from students' ways of coordinating sound- or word-images and representations of their experiences that are compatible with the coordinations constructed by other speakers of the language (Glasersfeld 1995). The

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discipline or subject matter may be different but the basic tenet of CT, to learn from and create models of students' ways of coming to know, is the same. Moreover, in school contexts where curricula are predetermined by educational authorities adopting a realist epistemology, and where teachers need to fit in the teaching of the subject content in the number of contact hours they have with their students in the span of a scholastic year, the (Subject)-Negotiation-Learner construct may be found effective to reflect upon and analyze teaching episodes with regard to the extent to which constructivist teachers *still* manage to engage in CT by negotiating a two-way road between the specific subject and the learner.

« 45 » Teacher apprehensions to implement CT as analyzed by M-N-L, such as attending to one student's conceptual operations at the expense of other students in the class, or giving each student time to express their thoughts knowing that time for the lessons is limited, was beyond the scope of this article. Nonetheless, research which analyzes the extent to which constructivist beliefs are implemented in lessons may also find the framework useful to investigate such difficulties, which may well be the reason why constructivist teachers may sometimes fail to act on their epistemological beliefs in their lessons.

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Open Peer Commentaries

on Philip Borg et al.'s "Negotiating Between Learner and Mathematics"

Investigating Teaching from a Constructivist Stance: A Model of Communication

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> Upshot • Borg et al. provide a framework that contributes to a growing body of research on how radical constructivism can help teachers and researchers to understand the complexity of classroom interactions. The bulk of my commentary is written to clarify theoretical points that I think are important to the endeavor that Borg et al. set out for themselves. My points come in part from thinking about how non-constructivist readers may interpret what Borg et al. outline in their article.

« 1 » In their target article, Philip Borg, Dave Hewitt, and Ian Jones outline a framework for examining constructivist teaching, making an argument for why they use the term "constructivist teaching" (CT; §1). I would prefer to see them consider their work to be an investigation of teaching from a constructivist perspective. The reason I would prefer this framing is that the term "constructivist teaching" can be interpreted to mean that a teacher might engage in teaching that is constructivist at certain times, and not constructivist at other times, depending on the actions that she takes. As a constructivist, I would prefer readers to understand radical constructivism (RC) as a theory that can help to provide explanatory accounts of teaching rather than as a label that determines whether and when a teacher might engage in a certain kind of teaching. I do not think that Borg et al. intend the "whether and when" connotation, but there are passages in the text that can be read and interpreted in this way.

« 2 » One example of a passage that can be read this way is the following,

> "These four elements show that deciding whether one's teaching may be called CT is not a simple dichotomous matter of determining whether the teacher takes or does not take into account learners' mental processes (§43)."

As part of this passage, Borg et al. identify four elements that they consider important to "constructivist teaching" (§42). For me, this quote, as well as others in the article, read as if a teacher might shift between constructivist and non-constructivist teaching based on certain actions that the teacher takes. Borg et al. note in this passage that these actions include more than just attention to learners' mental processes, but still the passage focuses on certain actions that a teacher might take that constitute constructivist teaching.

« 3 » A second example of this issue can be found in the way the research questions are asked, and the responses to them. In §15, Borg et al. ask:

> "What are the significant moments in the lessons where the teacher can be observed *trying to teach* in a way that is *sympathetic* to constructivist notions of learning? What are the significant moments in the lessons where a teacher can be observed to be *less sympathetic* to constructivist notions of learning?" (italics added for emphasis)

« 4 » To some readers, a response to the first question may signal that a teacher is engaged in CT whereas a response to the second question may signal that a teacher is not engaged in CT. Because Borg et al. use their framework to respond to both questions, it was clear to me that they did not intend this dichotomy. However, passages like the following appear when Borg et al. respond to the first question, "we attempt to demonstrate moments in different lessons when the teacher seemed to make such shifts of focus in order to engage in CT" (§19), suggesting that the shifts in focus somehow created occasion to engage in teaching that was constructivist. Similarly, passages like the following appear when Borg et al. respond to the second question, "data revealed that even a self-claimed RC teacher could create barriers to the negotiation of teacher and learner knowledge when he lost or ignored this sensitivity [to the learner]" (§36), suggesting (perhaps) surprise that a self-proclaimed radical constructivist could create barriers to the process of negotiating meanings.

« 5 » The point here is that a teacher (or researcher) with a commitment to the basic tenets of RC can investigate teaching *and* this investigation can be framed in terms of how RC, as a theory, helps to explain classroom interactions, rather than framed in terms of whether and when a teacher engages in "constructivist teaching." Using RC as a theory to create explanatory accounts of teaching seems aligned with the goals of Borg et al. Therefore: **Would Borg et al. consider using the term "teaching from a constructivist stance" rather than "constructivist teaching" (especially given that it may not convey their intended meanings to non-constructivist readers)? (Q1)**

« 6 » One theoretical construct that I think helps to make this point explicit and may help Borg et al. develop other aspects of their framework is Patrick Thompson's (2013) model of communication (see also Thompson 2000; Glasersfeld 1995).[1] Thompson defines communication as the assimilations and accommodations that two or more people make in the process of interpreting utterances, gestures, representations, etc. (communicative acts) that they attribute to another person (see Figure 1).

« 7 » Figure 1 is intended to show that person A has the intention of communicating particular meanings to person B. As part of this process, person A's communicative acts are based both on the meanings that she wants to convey and her model of how person B may understand her meanings. Person B then interprets person A's meanings both in relation to his own meanings and in relation to his model of person A, and formulates a response to person A. This model helps to make explicit a constructivist perspective on how communication between two (or more) people takes place, and, I think, could help Borg et al. maintain a focus on theoretical constructs that can be used to analyze student–teacher interactions rather than criteria that might be considered constructivist teaching.

« 8 » Thompson's model of communication makes explicit other issues that are relevant to the Borg et al. framework. For example, Thompson's model of communication follows from one of the basic tenets of RC: that a person cannot escape from her own ways of perceiving and conceiving. Thompson expresses this tenet of RC in his model of communication by noting that each person in Figure 1 is directing her communicative acts at her *model* of the other person rather than at the other person directly.

« 9 » The pertinence of this point can be seen in relation to how Borg et al. analyze the data excerpts that they present. For example, when discussing one of their participants, Omar, they state, "the ensuing dialogue suggests that the teacher was comparing Omar's MOS with the teacher's own 'mathematics' […]" (§23). Thompson's model of communication would suggest a modified account – that the teacher was in fact comparing his second-order model of Omar's mathematics to the mathematical meaning that the teacher intended to convey to Omar. Although Borg et al. are clear on this point in many places in the article (e.g., §3, §5), including right before the quoted passage, there are other places where the distinction fades away, particularly in the conclusion. For example, Borg et al. state in the conclusion that

> "[…]the framework places the teacher as a negotiator between the subject matter and the learner and suggests that the main task of constructivist teachers is to find ways of building roads between their subject content knowledge […] and that of their students." (§42)

Here the statement reads as if the teacher negotiates between "subject content knowledge" and "their students." A reframing of this statement using Thompson's model of communication would suggest that a teacher negotiates between her mathematical meanings (her understanding of the subject content knowledge) and the model she has of her students' meanings both in preparing for, and in the act of, communicating with students. For me, these nuances are important to maintain throughout the article because they are central to what RC offers in making an explanatory account of student-teacher communication. Namely, RC opens the possibility of analyzing the mechanisms of communication between two or more individuals where the processes and products of communication do not have an ontological status outside of the ones that the individuals in the interaction confer on them. **How, if at all, do Borg et al. see this point as relevant to their purposes? (Q2)**

« 10 » Thompson's model of communication also helps to make explicit the importance of identifying what mathematical meanings a teacher intends to convey, and what her models of students' mathematics are. Borg et al. define mathematics for students (MFS) to include both of these, but throughout the article, I thought more attention could have been given to both. For example, Borg et al. state that the goal of using

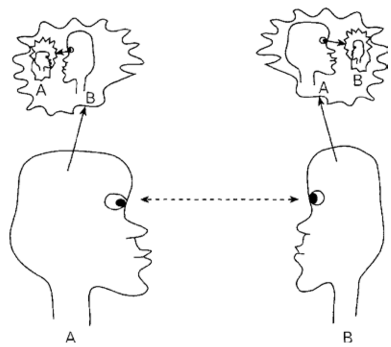

Figure 1 • Summary of the intersubjective operations involved in the communication of meaning (from Thompson 2013: 64)

Grid Algebra was "[…] due to its potential to help students conceptualise the meaning signified by standard mathematical notation […]" (§10). Here I wondered what meanings Borg et al. intended to convey for this notation. The data analysis section helped to identify some of these meanings (e.g., the difference between unknown and variable, that the equals sign can be thought about as representing a balance). However, unpacking the specific mathematical meanings the researchers wanted to work on with students and why they thought these meanings would be appropriate based on their models of students' mathematics would help to deepen the framing of their work and the subsequent data analysis. **What mathematical meanings did they intend to work on with students, and how were these meanings appropriate given their models of students' mathematics? (Q3)**

« 11 » One final issue that Thompson (2013) identifies in his model of communication is that as the process of communication unfolds, persons A and B adjust their understandings of the other person's understandings, and possibly (although not necessarily) their own understandings. A conversation between two or more people enters a state of intersubjective agreement when person A and B see no reason to think that they have misunderstood the other. As Thompson notes, this is not a claim that the two people share the same meanings or that they agree (i.e., inter-subjective agreement does not imply consensus). It is simply a claim that, for the time being, there is no reason for either

1 | Cf. also Patrick Thompson's unpublished presentation "Remarks on representations, conventions, and common meanings" at the panel for the PME-NA XXI working group on representations, Cuernavaca, Mexico, 1999. http://patthompson.net/PDFversions/1999Rep.pdf

person to think that they have misunderstood the other (Glasersfeld 1995).

« 12 » This issue is relevant to the Borg et al. framework in that it allows for the distinction between when a student makes an accommodation (i.e., act of learning) in relation to her understanding of the teacher's (or other students') understandings versus when a student makes an accommodation in relation to her own mathematical understandings. Both cases involve accommodation and may take place in concert with each other, but they can also occur independently.

« 13 » Throughout the data analysis I wondered whether and when students were making accommodations with respect to their understanding of the teacher's (or other students') understandings versus making accommodations in their own mathematical understandings. For example, in Protocol 3 I wondered whether the students had fundamentally changed their mathematical understandings in some way, or whether they had changed their understanding of the intended meaning of the teacher's question. This issue is important in characterizing the kind of learning that may be occurring in classroom interactions. In particular, it is instructive for teachers (and researchers) to consider how interactions among students and teacher could attain the status of inter-subjective agreement without students fundamentally changing their mathematical understandings. Finding evidence for this, and contrasting it with when students make accommodations in their own mathematical understandings is an important nuance that Borg et al. could address more explicitly in future work. **Do Borg et al. see this distinction in the data excerpts that they share? If so, how? (Q4)**

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Facilitating Constructivist Principles in Using Apps: Moving from Class Video to Community

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> Upshot • The target article offers a method for teachers to reflect on their constructivist approach in classrooms. This commentary suggests ways to augment the approach for use with groups of teachers.

« 1 » The authors of the target article used video recordings to analyse mathematics teaching with a view to facilitating analysis of one author (Philip Borg) using constructivist principles in class using a maths software programme (Grid Algebra). They described the tensions created by the need to teach a set curriculum to pupils who vary significantly in terms of their readiness (§9). The introduction refers to the debate about use of the phrase "constructivist teaching" since constructivism has at its centre the idea of learning. A first point for the authors is to ask them to extend their brief mention of the debate concerning constructivist teaching (CT). For example, the issue of the role of the teacher in constructivist learning is admirably illustrated in the target article (§1). **However, what are key differences they refer to between CT and progressive methods? And again, would the authors agree that while CT is not one particular method, it might be the case that individual teachers can be expected to develop their own styles of CT? (Q1)**

« 2 » The article shows in detail how the use of video facilitated the teaching author's awareness of his own strengths and weaknesses in his own classroom work. In what follows I want to discuss some implications I find in this work for helping teachers to engage reflectively with their own practices with their pupils.

« 3 » The authors' perspective in using Grid Algebra (GA) software begins with the framework of relations between mathematics, negotiation and learner (M-N-L) (§4). This raises the possibility of discussing the approach taken from the "teacher perspective" and in addition moving beyond this perspective to consider the social interactive level of the mathematics community in the classroom. The focus of the target article is principally on the M-N-L framework (§2) emphasising teacher-learner interactions rather than including learner-to-learner comments and interactions. **Would it not be useful to further analyse these videos to examine ways the teacher and pupils facilitated or participated in co-operative learning? (Q2)**

« 4 » One of the contributions of the target article is that it allows teachers and teacher educators to appreciate the complexity involved in balancing set curricula with fixed goals with widely varying pupil/student understandings of the material in an educational setting. This tension is clearly present in comments made concerning the teacher's difficulties maintaining a constructivist stance in class. In particular in (§§38f) where the teacher regrets not being able to pick up on pupil initiatives, it seems to me that there may be ways to acknowledge pupils' different perceptions without allowing these perceptions to derail the focus of the class conversation. On the basis of my experience observing and supervising student teachers in classrooms, I am aware how easy it is to miss opportunities to capture moments when individuals might have been helped, had the teacher noticed or acted. Yet, at the same time, classes need a flexible focus where the teacher can notice an opportunity and decide whether to stop or return to it later. **Perhaps the important thing is that the pupil knows the teacher noticed and that later the teacher follows up with the pupil? (Q3)**

« 5 » There will always be moments when teachers have to decide whether to follow the lesson plan or to stop and help individual students. Constructivist theory prioritises the moments when pupils show that uncertainty may be the opening to learning something new. Perhaps this is part of the art of teaching, knowing when to stop and digress and generate a context where lasting learning will occur.

« 6 » One question I had on reading (§§11f) concerns the issue of moving from understanding how to use the software to learn that multiplication is a form of successive addition, and conversely to learn division is a form of successive subtraction. I have a memory of a maths class when I was making this link with some difficulty! I knew how to divide and subtract, but there was another awareness that I was having difficulty constructing, namely the awareness that these different operations were connected. Today I might call this a higher-order connection. Aged about 10, I was missing this awareness that dividing 2 into 5 and obtaining the answer 2 with 1 remaining, was precisely the same as taking 2 from 5 successively (twice) and having 1 left over. So turning this memory into a question for the author, how do you avoid ensuring that pupils using GA software with row and column manipulations to find answers will also notice these deeper mathematical relationships between addition, multiplication, subtraction and division? Or putting it another way, is there a danger that the pupils will use the software like a calculator and not develop a feel for whether answers are correct or not? Is there a danger that pupils will not move from learning about the tool at a simple level to learning with the tool at different levels? So, would it be an advantage for teachers to design tasks for pupils to work in groups using GA to ensure these relationships are well understood and therefore ensuring the advantages of learning through social cooperation (Gash 2014: §27).

« 7 » If a group of teachers could arrange to view videos of one another's classes together and share their views this might be a productive way of promoting professional development. Recently in *Constructivist Foundations*, Karen Brennan (2015) discussed the evolution of teachers' skills using technology. While Brennan focused on constructionism and so on the values of learners designing and making things in the classroom, it seems to me that the social processes she described would be one excellent next step in the author's exploitation of the experiences described in the target article. Another way of using video to promote professional development is described by Jana Visnovska and Paul Cobb (2013). **Perhaps the authors are already engaged in these types of activities? (Q4)**

« 8 » The moments of uncertainty or frustration the author describes in the target article (§22) are moments that have been commented on in Therese Dooley's work (2010). Dooley also focussed on learner epistemic actions including recognising and constructing; these actions resonate with bridging between the learner and the features of negotiation (§18). These commonalities in the approaches of mathematics teachers coming to terms with the tensions between emphasising learning as constructivists and being responsible for set curricula demonstrate the importance of methods such as these that facilitate self-reflection. A strength of the approach in the target article is that the M-N-L tool offers a way of examining dynamic changes in both teacher approach and learner construction.

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What Do We Lose If We Abandon Constructivism?

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> Upshot • While I appreciate sensitive teaching approaches to students' learning mathematics using Grid Algebra software, I am unconvinced that the approaches described are constructivist in nature. To make further progress along the lines described by the authors a clearer articulation of its constructivist foundations is needed.

« 1 » In their target article, Philip Borg, Dave Hewitt and Ian Jones discuss the teaching and learning of mathematics using a software tool Grid Algebra. The teacher is also a researcher, collecting and analyzing data from his interactions with students. In the article, episodes of dialogue illuminate and provide insights into the teaching approach, the ways in which the teacher thinks about mathematics and the learning of his students, and his own approaches to promoting/supporting their learning. A fundamental tenet of the teacher's practice seems to be his *sensitivity to students*, the forms that this takes and the tensions that arise. The authors write that the teacher is working from a *constructivist* perspective (*radical* constructivism is mentioned) and they talk about the teacher's "sensitivity" to his constructivist principles. Therefore my commentary starts with the meanings of *constructivism*, *constructivist teacher* and *constructivist teaching,* which are exercised throughout the article, and how these relate to the goals, actions and reflections of the teacher as expressed through the episodes.

What constructivism is – and the meaning of constructivist teachers and constructivist teaching

« 2 » At the beginning of the article, the authors write that "constructivism is a theory of learning, not of teaching" (§1). I certainly agree with this. However they refer, without further elucidation, to Martin Simon (1994) and Arne Engström (2014), whereas I might have in mind alternative sources such as Paul Ernest (1991) or Ernst von Glasersfeld (1983). Would these four sources agree on the theoretical groundings for stating that "constructivism is a theory of learning not teaching" (§1)? I make the point here that the nature of the theory of constructivism, as a basis for this article, needs more than a passing reference (or two). Here, the reader is left to infer the authors' theoretical groundings. For example, readers are given some hints: a constructivist epistemology contrasts with a *realist* epistemology and is compatible with *progressive* modes of teaching in which "a second-order observer might consider [exposition] 'traditional' and 'non-constructivist'" (§5B). Unfortunately, the terms *realist* and *progressive* are not discussed either.

« 3 » In Jaworski (1994: 15–25) I explained in some detail my own understandings of constructivism and its relations to the learning and teaching of mathematics. On the one hand, if the constructivist epistemology of the teacher accords with *radi-*

cal constructivism, it might be seen to agree with Ernst von Glasersfeld's two maxims: (1) "Knowledge is not passively received but actively built up by the cognizing subject" and (2) "The function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (Glasersfeld 1989b: 162). This would beg questions about processes of building knowledge, adaptive cognition and organization of the experiential world in relation to the research reported. Despite this theory resting firmly within the province of knowledge and learning, von Glasersfeld, elsewhere, suggests a set of five noteworthy consequences of a radical constructivist philosophy for the *teaching* of mathematics (Jaworski 1994: 26f). On the other hand, if the teacher's constructivism accords with *social* constructivism, it might be seen to agree with Ernest's perspective that "[…] there is the essential role played by experience and interaction with the physical and social worlds, in both the physical action and speech modes" (Ernest 1991: 72). While it is possible to rationalise these two perspectives (they are not incommensurable),[1] the nature of the rationalisation cannot be taken for granted. The authors mention the work of John Dewey, Martin Simon, Heinz Steinbring and David Kolb, but no further indication is given as to which concepts from these theorists are valued particularly, or whether such concepts are commensurable in relation to the research focus of this article. **So in what ways do the authors' own constructivist principles (perhaps expressed in relation to Dewey, Simon, Steinbring and Kolb) relate to the interactions between teacher and students that are reported in the article? (Q1)**

The theoretical framework M-N-L

« 4 » The authors present their own framework, the M-N-L framework, based on the teacher's negotiation between the mathematics and the learner (the student – although, as is made clear, the teacher is also a learner) to theorise the processes that they see emerging from their analysis of classroom interactions. For me, this framework is well explained and holds credibility in relation to the teaching described and the issues raised for the teacher. I appreciate particularly the concepts MFS (mathematics for students, as conceptualised by the teacher) and MOS (mathematics of students, as justified by a carefully-observing participant and analyst). However, I wonder how MOS and MFS are linked to constructivism, as this is not made clear. I infer that it is to do with the concept of "second-order models" of MOS (§5; for "second-order models" see also Steffe 1999: 5). Since the teacher cannot experience directly the mathematical thinking of her students, she has to develop her own model of this, based on her observation and analysis; from this she is able to relate her perception to her own mathematical planning for the students and thus modify her practice if it seems appropriate. This fits very well with an interpretivist frame as I have described in Jaworski (1994: 65–68): there I cite Aaron Cicourel who writes,

> "When the observer seeks to describe the interaction of two participants the environment within his reach is congruent with that of the actors […] but he cannot presume that his experiences are identical to the actors […] The observer is likely to draw on his own past experiences as a common-sense actor and scientific researcher to decide the character of the observed action scene." (Cicourel 1973: 36)

It strikes me that this is precisely what the teacher is suggesting in his conceptualization of MOS. **So, in what ways do the authors see these concepts aligning with constructivism? (Q2)**

Sensitivity to students

« 5 » The authors write about the teacher's so-called "sensitivity to constructivist notions" (§8) and suggest the teacher's *loss of sensitivity* towards constructivist notions when one of the negotiation paths on the road between learner and mathematics is blocked (§17B). In line with comments above, I would like the authors to be clearer as to what exactly are these "constructivist notions." They write "All this would not be possible without the teacher's *sensitivity* to students' mathematical understandings" (§8) and refer to Jaworski (2012). The phrase "sensitivity to students (SS)" derives from my 1994 work, cited above, and is one dimension of a "teaching triad" in which the other two are "management of learning (ML)" and "mathematical challenge (MC)." In subsequent work (Potari & Jaworski 2002) sensitivity to students is characterized as *affective*, *cognitive* or *social*; effective teacher-student interactions are seen to occur when there is "harmony" between SS and MC within ML. In the target article, I see the teacher's "sensitivity to constructivist notions" (or the lack of it) to relate to aspects of his teaching in which his cognitive sensitivity is overshadowed by MC or ML so that harmony is not achieved. It seems to me, in reading this article, that this teacher is extremely sensitive to making sense of how his students perceive the mathematics so that he can respond to them effectively. **However, how can sensitivity be seen as a constructivist construct? (Q3)**

The lessons and use of Grid Algebra

« 6 » The M-N-L model is based on negotiation of the teacher between the mathematics in focus and the student learner. However, in these lessons we see another dimension in addition to mathematics, teacher and learner; this is the software that mediates the teaching-learning process in accessing mathematical concepts. The teacher seems to use the Grid Algebra effectively to challenge students mathematically and support their developing mathematical cognition. The relationships here are reminiscent of those expressed around notions of *the didactic triangle* (mathematics, teacher, student) and *the didactic tetrahedron* (mathematics, teacher, student, software).[2] If we see the software as one of the key elements of the interactions described, then it seems to me that the software should be brought into the M-N-L model. Rezat & Sträßer (2012) use an Activity Theory model to try to capture the complexity of the relationships they see in the classroom situation. Since Activity Theory is one of a range of sociocultural theories, which, according to Stephen Lerman (1996), are *incommensurable* with construc-

1| Jere Confrey claims that "[t]he theory of constructivism has always been social" (Confrey 2000: 13).

2| Cf. ZDM issue on "New Perspectives on the Didactic Triangle: Teacher-Student-Content," in particular articles by Simon Goodchild and Bharath Sriraman (2012), and Sebastian Rezat and Rudolf Sträßer (2012).

tivism, there would potentially be problems in trying to unify Rezat & Sträßer's model with the M-N-L model, which is claimed to be constructivist. Here, theories or theoretical constructs that are plausible in their own right start to give problems when juxtaposed. This is one of the reasons why it is important to be clear and specific as to what aspects of a broader theory pertain to a particular construct.

« 7 » I will end with another question for the authors: **Given that M-N-L seems to be an effective model in theorizing the negotiation of the teacher between mathematics and students, with sensitivity to students (albeit lacking specific attention to the Grid Algebra software), but with dubious links to constructivist theory, what would be lost by abandoning constructivism as an overarching frame? (Q4)**

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Toward a Model of Constructivist Mathematics Teaching

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> Upshot • My commentary has two general goals. First, I investigate how basic principles of radical constructivism might be used in constructing models of mathematics teaching. Toward that end, I found that I was not in complete intersubjective agreement with Borg et al.'s use of some basic terms. Second, I explore what mathematics teaching might look like in what is construed as constructivist mathematics teaching. Toward that end, I comment on Borg et al.'s use of "negotiation" and explain how a constructivist teacher can establish experiential models of students' mathematics.

« 1 » Philip Borg, Dave Hewitt and Ian Jones have chosen to address what has become a pronounced problem in outcome-based mathematics education, and that is the disconnection between the mathematics curriculum and students' mathematics. The approach the authors take to gaining insight into this problem is from the perspective of a practising mathematics teacher who attempts to use principles of radical constructivism (RC) in developing a guiding framework for engaging in constructivist mathematics teaching. The framework is based on the authors' meaning of several key terms, some of which I find problematic. So rather than discuss their framework per se, I focus on their interpretation of some of the key terms and consequences following on from the interpretations in order to engage in a discussion of constructivist mathematics teaching.

"Mathematics" and consensual domains

« 2 » The first key term, "mathematics," is taken as a consensual domain of mathematical concepts and skills existing among a group of persons. Because Ernst von Glasersfeld (1990b) spent considerable time interpreting Maturana, I once asked him how he regarded Maturana's idea of a consensual domain. His explanation was that,

> "A consensual domain is established when the individuals of a group adjust their actions and reactions to achieve the degree of compatibility necessary for cooperation. This involves the use of language and the adjustments and mutual adaptations of individual meanings to allow effective interaction and cooperation." (Personal communication)

By using the term "existing" in their account of "mathematics" as a consensual domain, the authors seem to regard "mathematics" as "out there" in-between or among the individuals of a group, whereas von Glasersfeld focused on mutual adaptations of individual meanings to allow effective interaction and cooperation. Positing the "existence" of a consensual domain of mathematical concepts and skills in-between the participants in an interaction is not inappropriate if "existence" is taken from the point of view of an observer who observes cooperating individuals in effective interaction. That is, it takes an observer to interpret the observed interaction and, on that basis, construct what she construes as consensual mathematical concepts and skills "existing" among the participants in an interaction. The observer may well be one or more of the individuals involved in the interaction who assumes the perspective of a second-order observer, which is included in Maturana's account of an observer.[1] In a case where the observer is a teacher, the teacher may construct what she construes as mathematics of students involved in the interaction that she imputes to the involved students. She may also decide where she might choose to take the students in subsequent interaction, which is what the authors refer to as the mathematics for the involved students.

Interaction

« 3 » I found the protocols and the discussion surrounding them consistent with the authors' account of interaction, which is couched almost exclusively on the teacher's side of the interaction, where the students'

1 | "Everything said is said by an observer to another observer, who can be himself or herself" (Maturana 1978: 31).

role is interpreting the teacher's utterances, etc. The authors also consider the principals involved in an interaction as observers of their own experiences made up of their own re-presentations and of the occurrences that follow those re-presentations, which they consider as first-order observers (§3). This characterization of first-order observers fits the concept of second-order observers in that it involves awareness of and reflection on regenerations of past experiences that presumably refer in this context to experiences of mathematics teaching. In an ongoing episode of teacher-student interactions, a mathematics teacher can stand back from the interaction and posit situations or tasks "on the fly," a phrase Edith Ackermann (1995) used to describe how hypotheses are formulated in clinical interviews. So, a teacher engages in second-order observation when involved in reflection on regenerations of her own past experiences of teaching students for the purpose of generating situations to engender what is regarded as desirable student activity. Apparently, the authors consider second-order observers as restricted to observing interaction between two other agents that do not include themselves (§3). In the one case where they considered the possibility of an observer observing him or herself, it was as researcher during the data analysis.

« 4 » The authors' account of interaction was basic in their concept of the teacher as negotiator between the learner and the subject content because there was little development of the teacher's model of, in the authors' terminology, the MOS (§4). There are three important distinctions in a constructivist teacher's negotiating between "mathematics" and students' mathematics that were not mentioned by the authors. First, a distinction can be made between a teacher engaging in responsive and intuitive interaction and engaging in analytical interaction (Steffe & Ulrich 2013). In the former case, the teacher is an agent of interaction [and action] who harmonizes herself with the students with whom she is working. She "loses" herself in the interaction and makes no intentional distinctions between her knowledge and the students' knowledge. At this point in teaching, the teacher is not an observer, but an actor somewhat like von Glasersfeld described, where "there is no split between the experiencer and what was being experienced" (Glasersfeld 2009: 257ff). Second, in the case of analytical interaction, the teacher "steps out" of her role in responsive/ intuitive interaction and becomes a first-order observer of her students' language and actions and focuses on analyzing their thinking in ongoing interaction (Steffe & Wiegel 1996). The teacher's interventions are focused on constructing experiential models of the students mathematics with whom she is working. In essence, she becomes the students and attempts to think as they do in analytical interaction (Thompson 1982, 1991; van Manen 1991). These attempts constitute ultimate acts of decentering on the part of the teacher and constitute constructions of meanings of interaction from the students' side of the interaction, which is an essential aspect of constructivist teaching. **How would the authors explain the students' side of an interaction? (Q1)**

« 5 » Finally, in that case where the teacher intends to investigate student learning, the teacher becomes a second-order observer. As a second-order observer, the teacher focuses on creating situations of learning that she might use to engender accommodations in students' ways and means of operating (Steffe 1991b) that explicitly as well as implicitly takes into account the mathematical knowledge of the teacher as well as the mathematics of the students (Steffe & Wiegel 1996). That is, as a second-order observer, the teacher establishes zones of potential construction (Olive 1994: 163; Steffe & Olive 2010: 17ff) for the students prior to interacting with them and, in the context of actual interaction, establishes actual zones of construction that can be used in the teacher's construction of further zones of potential construction.

Students' agency

« 6 » A major goal of the constructivist teacher is to interact with students in such a way that not only allows or permits them to become active agents of action or interaction, but also brings forth and sustains students' more-or-less independent mathematical activity and interactivity, which may transcend the teacher's suggestions or directives (Steffe & Wiegel 1996; Steffe & Tzur 1994). In reading through the text surrounding Grid Algebra, I didn't read protocols or passages that characterized the students in these ways. In contrast, the image portrayed was one where the teacher basically did not relinquish control in that the teacher seemed to be centrally involved in each protocol. **Simply put, had the teacher established the classroom in such a way that he could step out of the classroom for rather extended periods of time with the confidence that the students would remain deeply engaged in goal-directed mathematical activity or interactivity? (Q2)**

Experiential models of students mathematics

« 7 » Following the major goal identified in §6 above, another basic goal of a constructivist teacher is to construct experiential models of students' mathematics and to use those models in the construction of a mathematics for children. There is a major distinction between anecdotal accounts of mathematics of students' and what are considered experiential models. For example, in §26 the authors interpreted the tone of Jordan's comment as indicating that he did not consider negative integers to be multiples of two, like positive integers. Although I find the inference suggestive, extensive investigation in the context of teaching would be needed to explore the depth and breadth of how Jordan might operate in the context of constructing "negative integers" (Ulrich 2012). I have already commented that the bottom line in establishing experiential models is whether the teacher can think and operate as if they are the students, which implies that the students are intensively engaged in mathematical activity. Strictly speaking, a living, experiential model consists of one or more students' characteristic ways and means of operating in a consensual domain.[2] **So, how is a reader to interpret the anecdotal accounts of students' language and actions in the protocols? Did the teacher take them to establish consensual domains among the students? If so, was Dan's comment "The one-times table?" (§29) imputed to the other students in the class? (Q3).**

2| I appreciate that the author's may have had no intention to work with Jordan on negative integers.

« 8 » My use of the phrase "mathematics of students" is somewhat different from the way that the authors use it. I use "students' mathematics" to refer to the general assumption that students indeed have constructed ways of operating that could be thought of as mathematical simply because they are human beings not unlike myself. My use of "mathematics of students" is in reference to students' characteristic ways of operating that I have been able to bring forth in students as they engage in more-or-less independent mathematical activity and interactivity. In the authors' usage, the phrase "mathematics of the students" (MOS) refers to the uninterpreted student. The authors speak of models of the MOS but they do not use a phrase to refer to teachers' knowledge of the MOS, except for "second-order experiential models" (§3). By more specifically using "mathematics of students" to refer to second-order knowledge of students' mathematics, not only does it highlight reference to an experiential model a teacher may have constructed of one or more students, it also highlights the more-or-less independent mathematical activity or interactivity that serves in the construction of the experiential model.

"Mathematics curriculum" and mathematics of students

« 9 » Along with the authors, I separate the teacher's knowledge into two parts, the part that she would not attribute to her students and the part that she has learned by means of interacting with students; that is, in my terminology, the mathematics of students. For clarity, the teacher's knowledge of the curriculum can be considered teacher's knowledge that is complementary to (but perhaps overlapping) her conception of students' mathematics (Steffe 2007). Viewed in this way, that part of "the curriculum" that does not overlap her mathematics of students is just as inaccessible to the students as students' mathematics is to the teacher. Both are examples of first-order knowledge, which are models that subjects construct to organize, comprehend, and control their own experience (Steffe et al. 1983: xvi). The experiential models the teacher constructs to account for what students say and do constitute second-order knowledge (ibid). Likewise, the mathematics the students construct of the teacher's mathematics are the mathematics of the teacher from the students' perspective. So, the problem in mathematics education that the authors address, which is the disconnection between the mathematics curriculum and the MOS, can be recast as the disconnection between the teachers' knowledge of the curriculum and the teachers' knowledge of students' mathematics, which I have referred to as mathematics of students. Recasting the problem in this way conceives of the teacher as an intelligent and adaptive teacher capable of engaging in ethical teaching, and it is compatible with the authors' claim that they are not presenting a curriculum-versus-learner construct.

> "Rather, we propose that just as 'two points define a straight line' (Dewey 1902: 16) so does the 'mathematics' of the teacher and that of the student (MOS) define the course of teaching." (§4)

However, I do believe that the authors would agree with me that Dewey's metaphor is misleading in that the relationship between the students' construction of elements of the teacher's knowledge of the curriculum and the teacher's construction of students' mathematics does not proceed along a linear path.

Final comments

« 10 » The authors opened their article with the question concerning whether there is a kind of teaching called "constructivist teaching." The claim that RC is a theory of learning but not teaching (Simon 1994; Engström 2014) is, in my opinion, a misconstrual because RC is a general model of knowing (Glasersfeld 1995) that has been used in family therapy and other fields. It does contain the constructs assimilation, accommodation, and equilibrium, and these constructs can be used in constructing models of specific aspects of mathematical learning. But the constructs can be used in establishing intersubjectivity among the participants in conversation as well (Thompson 2013). Consequently, we should not attempt to simply *apply* RC to mathematics learning or to mathematics teaching in an attempt to improve either. Rather, mathematics educators must engage in the hard work of constructing models of mathematics teaching that are similar to the available explanations of students' mathematics using the tools of the general model of knowing that are grounded in the experience of teaching students. It is in this context that I agree with the authors that it is the teacher who must negotiate between the teacher's knowledge of the curriculum and the teacher's knowledge of the mathematics of students. In fact, I regard the disconnection between these two aspects of teachers' knowledge as the most fundamental problem of mathematics education today and it is where mathematics education is failing worldwide (Steffe 1996).

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Negotiating the Classroom

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> Upshot • Borg et al. argue that there is a Mathematics-Negotiation-Learner (M-N-L) structure that can be used as a conceptual framework in order to evaluate the application of radical constructivism in teaching. This structure assumes a coherent consensual domain that is the mathematics being negotiated. However, there are at least four different consensual domains that make up mathematics. The mathematics that is the consensual domain in an RC classroom has distinct features that are designed to support the student's construction of mathematics.

« 1 » In their target article Philip Borg, Dave Hewitt, and Ian Jones argue that there is a Mathematics-Negotiation-Learner (M-N-L) structure that can be used as a conceptual framework in order to evaluate the application of radical constructivism (RC) in teaching. They do an excellent job in targeting the language they are using by careful definitions of these terms. The authors conclude that M-N-L is an effective instrument that helped in determining the extent to which the teacher was sensitive to his own beliefs. While I agree with the authors, in my judgment, the argument falls short by not examining the nature of the negotiation in an RC classroom.

« 2 » In §3 the authors argue that mathematics "is the consensual domain […] of mathematical concepts and skills existing among persons […]." The problem with this definition is that there are multiple consensual domains (CD) that make up mathematics. So what is the CD that is created by an RC teacher? In Richards (1991: 15f), I argued there are (at least) four different CDs in mathematics:

- *Research math* is the spoken mathematics of the professional mathematician and scientist.
- *Inquiry math* is used by mathematically-literate adults, including some mathematics teachers.
- *Journal math* is the language of mathematical publications and papers.
- *School math* is the discourse of the standard realist classroom in which mathematics is taught.

« 3 » Research math and inquiry math are structured according to what Karl Popper (1959) called a "logic of discovery." Journal math is based on journal articles that are designed to transfer information to a community that has already accepted many presuppositions – in our terms, the community is operating with an established CD. School math is patterned after journal math. Both school math and journal math are structured as "reconstructed logic" (in the sense of Popper 1959).

« 4 » The authors define mathematics for the students (MFS) as "[…]the mathematics the teacher intends to teach to a particular group of students" (§3). As it stands this definition is ambiguous between inquiry math and school math. Inquiry math emphasizes the discovery (or rather, the construction) of mathematics. For the teacher and student this involves the negotiation of a CD. School math, as a reconstructed logic, assumes the existence of an established CD. In point of fact, this explains some of the core problems with applying the realist perspective to a classroom. When a student is learning there is a natural negotiation of an evolving CD. In Michael Oakeshott's words, education is an initiation into the conversation "begun in the primeval forest and extended and made more articulate in the course of centuries" (Oakeshott 1962: 199).

« 5 » The challenge for the RC teacher is to support a logic of discovery in the classroom and to promote the student's exploration of mathematics. This leads naturally to a discussion of the nature of the conversation in a mathematics classroom. As Alan Schoenfeld argues, the problem is that teachers are telling students about problem solving. For Schoenfeld, "the primary responsibility of mathematics faculty is to teach their students to think: to question and to probe, to get to the mathematical heart of the matter" (Schoenfeld 1983: 2).

« 6 » The student's construction of mathematics results from an active exploration of, in this case, problem solving. The primary distinction is between the teacher "telling" and the student constructing. The teacher supports construction by posing problems, asking questions, discussing, and suggesting counter-examples in the context of a genuine mathematical discussion. As the authors point out in the lead quote from Paolo Freire, both the student and teacher are learning and teaching. That is because in a genuine mathematical discussion both the teacher and the student are listening.

« 7 » In the article, the teacher-student negotiation is supported by the *Grid Algebra* software. The software is designed explicitly for classroom use to promote exploration. It contributes to establishing classroom culture by structuring and constraining the conversation. Note that the software does not correct the student, it behaves. It is a puzzle that the student acts to solve. In an important sense Grid Algebra is part of the classroom negotiation – involving both the teacher and the student. In less experienced RC teachers it provides a scaffolding for the classroom discussion.

« 8 » The M-N-L structure is indeed useful for evaluating the constructivist teaching within a school context. But, in my judgment, it can play a more powerful role when we analyse the nature of the mathematics that is being negotiated. The nature of inquiry math provides guidance for teachers who want to create a climate of exploration, and it supports teacher preparation that emphasizes the establishment of a positive classroom culture that supports negotiation:

> "In class, try to avoid telling your students any answers […] Confront your students with some sort of problem which might interest them. Then allow them to work the problem through without your advice or counsel. Your talk should consist of questions directed to particular students based on remarks made by those students." (Postman & Weingartner 1969: 194)

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When Is a Constructivist not a Constructivist?

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> **Upshot** • I review the arguments put forward by Borg et al. as to why a teacher cannot be constructivist in their methodologies and ask why they have not considered constructivist methodologies that emphasise negotiation.

« 1 » Philip Borg, Dave Hewitt and Ian Jones present the Mathematics-Negotiation-Learner (M-N-L) framework, which they propose as a theory about what constructivist teaching (CT) *might* (my emphasis) mean, by "focusing on the role of the teacher in the educative process" (§4) and portraying her as a "negotiator" between the learner and the subject content (hence the dashes in M-N-L). The key concept within the framework is "negotiation," which I deal with especially in this commentary, but first, I must refer to the four primitives that Borg et al. forefront at the outset.

« 2 » The authors list four arguments to contest the existence of a kind of teaching that may be called "constructivist teaching." The issue of whether constructivism is or can be a teaching methodology or merely an epistemology has been argued extensively elsewhere (notably in Tobias & Duffy 2009), however, I think there are serious problems in authenticity in teaching when methodology is separated from epistemology. It is something akin to the "saving of appearances" standpoint from the history of science when Copernicus devised the heliocentric system but claimed it to be only a mathematical construct in order not to detract from the metaphysical worldview at the time. An epistemology that does not have a means to actualize itself is redundant, and a purported constructivist teaching methodology that is detached from authentic constructivist philosophy is at best "constructivism lite" or a kind of crypto-behaviourism. Certainly claiming to be constructivist in teaching but not in epistemology is contradictory.

« 3 » Referring to Martin Simon (1994, cited in §1), the authors list one of the arguments contesting the existence of CT as "Irrespective of [some] teaching approach[es], [some] learners will come to know by actively constructing mental constructs for themselves [some of the time]." Their argument appears to suggest that children will learn what they learn regardless of direction from teachers, and that what is needed is a minimal tweaking of the direction that construction takes place. The alternative in this paradigm would be a "teaching to the middle" or "educating some" paradigm.

« 4 » However, teaching approach does matter, as David Kolb (1984) would attest. Teachers must be clear about who should learn what, and why. Children construct idiopathically, and therefore the negotiation cannot be with the whole class but between small groups of learners, Constructivism seeks to move beyond "teaching to the middle" with the associated so-called differentiation: a little something harder for the smart kids and a little something easier for the others. **So, I must ask, what does it mean to Borg et al. to "know how to come to know?" (Q1).** To know how to come to know is to understand the process of learning itself, which is undoubtedly linked to exhibiting some understanding of the nature of knowledge. A mechanic must understand the process of a running car in order to repair a part or understand if a genuine/spurious part will suffice or even if doing nothing will affect the running of the car. This is dynamic knowledge, and a teacher must have something to say about the mind and the construction of concepts and their integration, hence the relevance to the work. Ernst von Glasersfeld (1989a: 123) stated that "'to know' means to know how to make." In other words, epistemology ("the what of knowing") is deeply embedded in methodology ("knowing how to come to know"). The big problem with CT is that the individual interaction it demands between teacher and student, or at best between small groups and the teacher is time-consuming – and the problem can only be solved by the quality of the teacher's ability to engage in small-group discussion in a whole class setting.

« 5 » CT may indeed be mistakenly equated with progressive modes of teaching (e.g., Engström 2014 cited in §1) such as "active learning," "problem solving," or "enquiry learning" but such a mis-equating of ideas is due to CT *not* being promoted as a progressive mode of teaching itself, with teachers and student teachers left to work out for themselves what works as a CT environment.

« 6 » Finally, Borg et al. believe a constructivist belief does not translate itself into a particular teaching method (Simon 1994, cited in §1), but to be constructivist it must still adhere to Ernst von Glasersfeld's two premises of constructivism.

« 7 » Borg et al. agree with all these arguments but still claim that the notion of CT is viable if it is attributed to a "teacher's sensitivity to learners' active and subjective construction of knowledge" (§2) during the teaching-and-learning process. So, if the child is a constructivist learner, the teacher can allow or tolerate it but not necessarily be a constructivist teacher – in other words, does it make sense to advocate constructivist learning without fully constructivist teaching? While Paul Kirschner, John Sweller and Richard Clark (2006) assume that learning is an individual process of knowledge acquisition, David Jonassen (2009) assumes that learning is seldom accomplished individually. According to Jean Lave and Etienne Wenger (1991) humans share their meaning and co-construct reality in communities of practice. Similarly, Marlene Scardamalia and Carl Bereiter (1991) used the phrase "communities of learners." Jerry Suls and Thomas Wills believe humans to be social animals who rely on feedback from peers to "determine their own identity and the viability of their personal beliefs" (Suls & Wills 1991: 116) such as affirmation within a community; therefore learning is socially negotiated.

« 8 » Socially-negotiated learning relates to concepts of metacognition according to the "five pillars of wisdom" of the general cognitive acceleration paradigm (Shayer & Adey 2002; Adey 2008). The Cognitive Acceleration through Mathematics Education (CAME) Project states the "five pillars" listed as follows with my own elaborations:

a *Social construction* of knowledge upon prior learnings, which can be sharing explanations and understanding of a problem and potential solutions.

b *Bridging* strategies with everyday experience, which can be working together to apply ideas "generated" in the lesson to problems in the world we experience.

c *Cognitive conflict* – challenging certainties, which can be thinking about a problem in a way that challenges prior knowledge.

d *Metacognition* – the explicit cognizing of students' constructions, which can be reflecting on thinking and articulating approaches to solving a problem.

e *Concrete preparation* of the cognitive "space" to determine "readiness" and which can be introducing a problem and helping with any new vocabulary or ways of doing.

« 9 » Negotiation is a feature of democratic and non-authoritarian personal interactions, as is metacognition. Social construction also involves negotiation and cognitive conflict can only be resolved satisfactorily through negotiation. Therefore, it is difficult to understand the decision neither to refer to the CAME Project, nor independently refer to metacognition or cognitive conflict. Therefore, this gives rise to a second question: **Why did Borg et al. not refer to or consider the CAME Project (Adhami, Johnson & Shayer 1995; Adhami, Robertson & Shayer 2004; Adhami, Shayer & Twiss 2005), or the wider implications of the cognitive acceleration paradigm? (Q2)**

« 10 » In the learning sciences, negotiation is best thought of as the means of seeking the student input into their own learning to arrive at a shared understanding. However, in the constructivist paradigm, it is not merely "tricking" the student to arrive at the same conception as the teacher by the artistry of teaching. In the constructivist paradigm, the teacher devolves their control by declaiming the ultimate authority of the subject content material itself, and thus the goal of mathematics education is not to acquire the mind of the teacher / mathematician, but rather develop the constructions of the student to be what they can be. The CAME project uses negotiation to assist learning in one or more, but at times all of the methodologies listed above. The thing is, negotiation, if authentic, does not try to steer a course to a direct goal known or pursued by one party to the negotiation. In constructivist learning, and teaching, negotiation will not be trivial, or inauthentic, but demand an openness that goes beyond mere sensitivity and makes the teacher as accountable to the learning contract as the student.

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Mathematical Observers Observing Mathematics

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> Upshot • Suggestions are made for ways of taking advantage of Borg et al.'s reference to the notion of observer for data analysis in mathematics education research.

"Thus, if someone claims to know algebra – that is, to be an algebraist – we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim." (Maturana 1988a: 4f)

« 1 » In order to ground their idea of Mathematics of the Students (MOS), in their target article Philip Borg et al. make use of second-order models and the notion of observers to discuss issues of data analysis. In this commentary, I suggest that these ideas can be further deepened by making reference to Humberto Maturana's theory of the observer. In Proulx (2014) I argue that some aspect of Maturana's observer[3] could lead us beyond, and even question, Leslie Steffe's use of second-order models to describe the mathematics in students' minds. From an observer's point of view, MOS does not "exist" in itself in the students' minds, but emerges and is distinguished by observers through the act of observing. Therefore, MOS lies in the eye of the observer, and what is recognized as mathematics is what the observer recognizes as mathematics on the basis of her own experiences that she understands as mathematical.

« 2 » Henceforth, we can re-use Borg et al.'s views of what the curriculum is for conceptualizing MOS, where MOS could be seen as something that observers have construed for themselves:

"RC teachers, however, believe that no such thing [i.e., curriculum/body of *a priori* mathematical knowledge] exists and that the curricular topics they are required to teach are actually knowledge they (the teachers) have construed for themselves." (§3)

« 3 » This view of MOS through the observer has quite some potential at the epistemological level for mathematics education research in relation to data analysis, as I already claim in a previous commentary (Proulx 2014). Here, I build on this commentary and re-insist and extend some of these ideas by adding concrete examples taken from my studies.

Observing possibilities

« 4 » In Maturana's theory of the observer (e.g., Maturana 1987, 1988a, 1988b), the observer is central to any *account* of any given phenomenon, for "everything said is said by an observer to another observer that could be himself or herself" (Maturana 1988b: 27). As explained in Maheux & Proulx (2015) and Proulx (2014), even if, as researchers in general, we may not believe that "the phenomenon being observed" is a fact independent of the observer and that it can be decontextualized from the observational act, we often take this position implicitly by how we report our findings. As Richard Barwell (2009) explains, even if we agree that we cannot account for what "really" happens, research is still being reported (and maybe even conceived) as if this were the case. This is, for example, the case when

3 | I do not pretend to represent, make use or apply Maturana's ideas of the observer here. The issues I raise have been inspired and occasioned in relation to his work. Thus, there is more in his writing than what I offer here, but also less. But, nonetheless, his writing has made these distinctions possible for me as a researcher.

it is suggested that a phenomenon analyzed from various perspectives "generates a *deeper* or *more comprising understanding* of the [observed] phenomenon" (Bikner-Ahsbahs & Prediger 2006: 56, my emphasis; see also Prediger, Bikner-Ahsbahs & Arzarello 2008). Or when researchers claim that the complexity of, for example, the classroom, cannot be *understood* only from one perspective (see, e.g., Even & Schwarz 2003). This position entails an assumption that "the phenomenon being observed" is a fact independent of the observer and can be decontextualized from the observational act. Therefore, not only can we look at the same data or the same situation from various perspectives, but we can also aspire to an "accurate account" of it.

« 5 » Through being attentive to Barwell's critique, the issue of "accurate account" is meaningless if the researcher holds an epistemological position where one does not *describe* what is being observed, but *constructs* one's own account of one's own perceptions. The adequacy of mathematical actions and strategies is not linked to some allegedly objective referent, but to the eye of the observer who *assesses* it on the basis of her own set of criteria: analyzing data rests no longer in their "truth" or validity, but in what they offer to us and to others.

« 6 » This observer's point of view transforms assertions about what are seen as research findings or MOS, and what can be learned from them. It proposes a transformation of view, not one focused on a state of affairs ("it is" *versus* "it is not") but toward the possibilities, the potential of these actions, to where they lead. In that sense, it is aligned to the mathematician Dave Henderson's view of mathematical correctness:

> "I relate correctness to the goal by saying that something is correct to the extent it moves an individual or group of individuals in the direction of an expanded understanding and perception of reality. […] How do we view mathematical arguments? When do we call an argument good? When do we consider it convincing? – When we're convinced! – Right? – When the argument causes us to see something we hadn't seen before. We can follow a logical argument step by step and agree with each step but still not be satisfied. We want more. We want to perceive something." (Henderson 1981: 13)

« 7 » This view is oriented toward what can be made possible by mathematical actions, or Borg et al.'s MOS, of where and what it can lead to, where it extends; rather than a focus on what it *is*, a state of affairs of what one does or does not know. The observer offers thus an analysis that develops understandings of MOS in terms of their potential, their mathematical possibilities and extensions.

« 8 » With this positioning, the stakes of analyzing data rest no longer in the "truth" or validity of students' mathematics, but in what they offer to oneself and one another. Thus, the approach engaged in for data analysis implies the *necessity* to move away from questions about mathematical knowledge or knowing and focuses on students' actions *for imagining possibilities for mathematics education*, for seeing extensions, rather than arguing for or against taken-as-given practices, activities, tools, and so forth. Following Simon Jarvis's (2004) idea of speculative thinking, the intention is to imagine possibilities, to draw them out by analyzing students' mathematics.

« 9 » Even if it prevents us from making direct assumptions about what students might know or not know – as if they were holding knowledge one way or another – studying students' mathematical actions makes it possible to make sense of these propositions as diverse ways to *approach* and *go about* students' mathematical activity or MOS. That is, regardless of the "trustworthiness" of the students' oral account of their thinking processes, or the possible relation of the strategies observed with fixed/preexisting forms of knowing that one would recognize and relate to, these strategies can be discussed in terms of *action possibilities*.

« 10 » As a way of illustrating these ideas about data analysis in terms of mathematical possibilities, I offer here an analysis of two examples taken from my studies on mathematical problem-solving.

Example 1: Operations on functions

« 11 » In one study, Grade-11 high-school students (15–16 years old) had to solve graphically usual tasks about operation on functions (Proulx 2015a). The graph of two (or three) functions was represented in the Cartesian plan on the whiteboard, and students had 15–20 seconds to operate on these functions and then draw their response on a blank sheet with a Cartesian graph on it (with the line $y = x$ drawn on it as a referent). Figure 1a illustrates a task where students had to add functions f and g, and Figure 1b displays one of the strategies students developed to solve it. To solve the $f + g$ task, students paid attention to the following points:

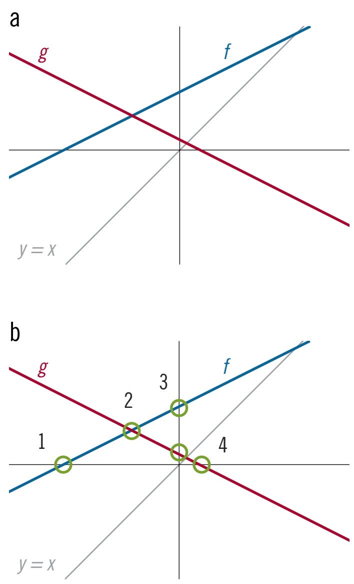

Figure 1 • (a) The $f + g$ task; (b) The strategy focused on significant points to solve it.

- where f cuts the x-axis (x-intercept), resulting in an image-length in g (because the image-length in f is 0);
- where f and g cross each other and have the same image-length, resulting in an image double the value of where they intersect;
- where f and g cut across the y-axis (y-intercept), resulting in a similar process as in (2);
- where g cuts the x-axis as in (1).

« 12 » This strategy, i.e., focusing on significant points that enable one to determine where the line/function is, has potential, e.g., in relation to an extension toward multiplication of functions. Indeed, paying attention to points 1 and 4 permits one to evaluate the general shape of image-length for an x smaller than that at point 1: with negative values of f multiplied with positive images of

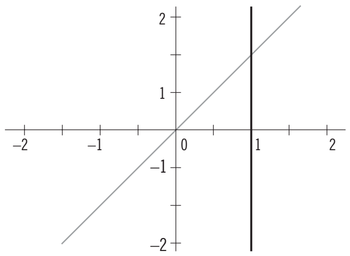

Figure 2 • The line $x=1$ given as the answer to the system "$y=x$ and $y=-x+2$".

g giving negative values in its multiplication; the same applies to images with an *x* bigger than the one of point 4. For values in *x* situated between points 1 and 4, the multiplication of images gives a positive value, leading one to see the quadratic function (second degree) created by the multiplication of two linear functions (first degree). Through the use of significant points this strategy reveals graphical aspects related to these functions, and this can also be linked to the study of zeros of functions and inflections points in calculus (e.g., here obtaining [+|−|+]).

Example 2: Systems of linear equations

« 13 » In another experiment, adult solvers (high school teachers) were given 15–20 seconds to solve a number of usual tasks on systems of equations, given algebraically on the board, and then to draw their response on a blank sheet with a Cartesian graph on it (also with the line $y=x$ drawn as a referent) (for more details see Proulx 2015b). In the case of solving the following system of equations "$y=x$ and $y=-x+2$," Figure 2 shows the answer given by one participant, the line $x=1$.

« 14 » This participant drew the vertical line, that is $x=1$, explaining that he did not have enough time to find the value of *y*, but that the solution had to be on this line because when replacing $x=1$ in each equation, it gave the same value. Note that the substitution of $x=1$ in the equations directly gives the value in *y* (equations being of the form $y=\mathrm{m}x+\mathrm{b}$). However, in his algebraic manipulations to find the value for *x*, the emphasis is on finding a common *x* that gives the same answer ($x=?$ and $-x+2=?$) and not on finding the value for *y* even if *it is* the same value. But in his strategy, both were done/seen separately.

« 15 » Even if incomplete, this strategy introduces many mathematical possibilities and extensions. The line $x=1$ permits representation of all possibilities for *y* to solve the system, even if only one value will be able to satisfy both equations simultaneously. In addition, this line $x=1$ represents a family of solutions to the system of parametrical equations "$y=\mathrm{x}+k$ and $y=-x+(2+k)$," leading to the study of parameters with $k=0$ for the parameter of this system that has (1, 1) as a solution. Also, this "omitting" to pay attention to *y* highlights the interesting obvious fact that the value for $x=1$ for both equations is the value for *y*, thus also working on the value in *y* because the value in *y* needs to satisfy both lines of the system of equations. Finally, the intersection of $x=1$ and the referent line $y=x$ is exactly where the solution point is situated. This makes it possible to insist that the solution *is part of* both lines of the system; a fact that is often overlooked when solving algebraic tasks.

Conclusion

« 16 » As the above analysis shows, the orientation taken toward MOS aims at bringing forth its potential and possibilities in a flexible and developing form rather than as a state of affairs as if it were static or a fixed form of knowledge. To consider it static would entail that mathematics or MOS exists in itself in the minds of students, a position contrary to that which I have developed here. Rather, inspired by Maturana's theory of the observer, I argue that MOS emerges in the eye of the mathematical observer (when) observing mathematics.

« 17 » As mentioned in Proulx (2014), the observer transforms the assertions about what are seen as findings and what is learned from them: pointing to potential extensions, suggesting what could be created by it and inviting us to think in alternative or even "futuristic" ways. Thus when we speak of mathematical possibilities, it is always from the point of view of the observer, in what this observer sees as mathematically possible.

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Authors' Response

The M-N-L Framework: Bringing Radical Constructivist Theories to Daily Teaching Practices

Philip Borg, Dave Hewitt & Ian Jones

> **Upshot** • We seek to address several questions and statements made in the commentaries by elaborating on the four main aspects of the M-N-L framework. Before doing so, we discuss the issue of constructivist teaching in the context of schools. We conclude by hypothesizing on what would be lost in the M-N-L framework by taking constructivism out of the picture.

Teaching from a constructivist stance

« 1 » Over the years, constructivist literature has included debates about the legitimacy of the term "constructivist teaching" (CT). In our target article, we attempted to partly reconcile opposing views by proposing that CT may be attributed to teaching with a sensitivity to constructivist notions. We described teachers' *actions* in the classroom incited from this sensitivity by the Mathematics-Negotiation-Learner (M-N-L) framework that was used by the first author[1] to analyze whether, when, and how CT was taking place in his own mathematics lessons as a school teacher. **Erik Tillema** asks whether it would be better to consider changing the term CT to "teaching from a constructivist stance" (Q1), which is more or less what Martin Simon (1995) refers to as pedagogy from a constructivist perspective. The meaning we give to CT does include teaching from a constructivist stance but it is not only that. Building on the work of Leslie Steffe (e.g., 1991a) we regard CT as also involving teachers' attempts to construct second-order experiential models of their students' knowledge and to let those models affect their own content and pedagogical knowledge. The process of building a two-way road connecting teachers' knowledge and these models of students' knowledge is our concept of "negotiation" in M-N-L.

« 2 » CT is inherently student-centred because it is wholly dependent on whether students are motivated and enabled enough to take an active part in the lesson. It is only by carefully observing students' communication and activities that teachers can build experiential models of their students' conceptual structures. Being student-centred, CT is often confused with progressive methods of teaching like problem-solving, discovery, and inquiry, which espouse a hands-on, active student participation. The overarching philosophy behind such methods is that teachers need to acknowledge that learners have a mind of their own and should allow them to take control of their own learning. Such was the philosophy of John Dewey, who began his campaign for a more active and self-directed style of learning over a century ago. The progressive movement that Dewey (e.g., 1902) advocated calls for teachers to link the knowledge they intend to teach to their learners' "experiential worlds" (Glasersfeld 1990a: 22), something which today may be identified with radical constructivism (RC).

« 3 » In response to **Hugh Gash**'s Q1 about our claim that CT should not be equated with progressive modes of teaching (target article, §1), we clarify that progressive modes of teaching might well have a CT approach but that CT does not necessarily imply a progressive teaching method. Our claim is that traditional teaching methods (which are sometimes carelessly dubbed as "teacher-centred") and CT are not necessarily mutually exclusive. We agree with **Gash** that CT does not translate itself into one particular teaching method and individual teachers may adopt different methods of engaging in CT. In fact, the teacher in our study preferred to use a mixed-methods approach. For example, in the same lesson, one could observe teacher exposition, class discussion, group work, inquiry, and problem-solving activities. If a third-party observer were to enter the classroom and observe the teacher during a teacher exposition episode, that observer could mistakenly deem that as a teacher-centred and pouring-in method of teaching. We attempted to show that even during such instances the teacher was engaging in CT if and when he was sympathetic to constructivist notions of coming to know and strive to negotiate paths between his knowledge and the models he was building of his students' knowledge.

« 4 » The main factor that determines whether teachers engage in or disengage from such a negotiation is their sensitivity to students (SS), in particular to the way students seem to be making sense of the classroom experiences or learning offers (Steinbring 1998). **Barbara Jaworski** Q3 questions how such sensitivity can be seen as a constructivist construct. SS is a teaching characteristic in **Jaworski**'s (1994) "teaching triad," a descriptive model derived from studying the way a classroom teacher, Clare, went about her daily business of teaching mathematics. Clare exhibited SS by her efforts to become aware of her students' conceptual structures, thinking styles, and tendencies, but not only that. Her SS was particularly evident when she was observed striving to make her learners feel respected, included, and cared for in such a way that they felt at ease communicating to her what they were thinking. In the M-N-L framework SS plays a very important role. If teachers do not have SS or choose to disregard it, they cannot create an atmosphere of interaction and communication of ideas with the students and the "negotiation" paths from "mathematics" to the learner will be obstructed, as we attempted to show by Protocols 5 and 6.

« 5 » In our understanding of CT, teachers need to "empathize" with their students before, during, and after interacting with them, especially when they seek to build second-order experiential models of students' knowledge. This "empathy" needs to be affective before it can be conceptual. In Protocol 1 (target article, §22) the teacher *sensed* that Omar was frustrated (Omar had let out an exasperated sigh) because he was getting an error message from the software. The teacher *acted* on that sensitivity and asked Omar, "What's the matter there, mate?" His use of the word "mate" or "pal" (in Maltese, "*xbin*," pronounced "*shbeen*") was an attempt to establish an affable, same-level communication which seemed to make Omar comfortable to talk about his difficulty in achieving his goal. Without establishing this friendly, caring rapport it would

1| The first author was the teacher-researcher in this study. He will henceforth be referred to as "the teacher have been unlikely for the teacher to form a model of the mathematics of the student (MOS) and identify the mismatch between this MOS and the "mathematics" coded in the software (which was the teacher's mathematics – the mathematics he wanted to teach). During this protocol the teacher

a was *affectively* sensitive to Omar's frustration/exasperation,
b acted on his affective sensitivity (started the conversation in a casual, friendly manner),
c was *cognitively* sensitive to Omar's mathematics, and
d acted on his cognitive sensitivity by asking questions to enable him to construct a model of Omar's conceptual structures.

So to answer **Jaworski**'s question, rather than being a constructivist construct per se, SS is a necessary characteristic for teachers to engage in what we are defining as CT. Whether teachers engage in CT depends on whether they decide to *act* on that SS. It also depends on their outlook towards the subject matter, in our case mathematics. In the following section, we elaborate on our definition of "mathematics," mostly in response to **Steffe**'s §2.

"Mathematics"

« 6 » Since we defined "mathematics" as the consensual domain (CD) existing among a group of persons, it seems appropriate to elaborate on our understanding of CD and the verb "existing," given **Steffe**'s (§2) objections to how we use these terms. Ernst von Glasersfeld argues that based on their different, subjective views of their experiential worlds people

> "can build up a consensus in certain areas of their subjective worlds. Such areas of relative agreement are called 'consensual domains,' and one of the oldest in the Western world is the consensual domain of numbers. The certainty of mathematical 'facts' springs from mathematicians' observance of agreed-on ways of operating, not from the nature of an objective universe." (Glasersfeld 1991a: xvi)

« 7 » Anyone who has constructed and developed some mathematical *concept* is a kind of "mathematician." Such concepts include those that do not make use of agreed-on conventions, like the concept of quantity (you don't have to know about numbers to make decisions about which tree is taller, which bag is heavier, which handful of sweets contains more…). Therefore, there is a global mathematical CD among members of the human race: any group of communicating persons would agree that a 20 m building is higher than a 10 m one, even if they never learnt about the quantities 20 and 10. However, within this CD of "mathematicians," people have invented, used, and passed on mathematical *conventions* with which the sense-making of their experiential worlds became more efficient and viable. A quick glance around us shows the extent to which these conventions are applied to enable us to cooperate in our search for explanations of our experiential world. Philip Steedman (1991: 7) says that mathematics is "a social creation which changes with time and circumstances." But as long as the mathematical concepts and conventions serve their purpose to help humans cope with their experiential worlds, they prevail, and for that purpose it is deemed necessary that they be included in school curricula in the form of branches, topics, and sub-topics. It is then up to teachers to *interpret* the "mathematics" written in the curriculum and re-present it through actions in the classroom, with the aim of bringing it "within reach" of their students, to acquaint them with conventions and concepts which may prove viable to explain their experiential world. **John Richards**'s (§2) distinction between *inquiry math* and *journal math* is very pertinent here. Teachers need to get to the heart of the topics in the *school math* written in their syllabi and convert it to *inquiry math* by giving their students the opportunity to be "actively engaged" (Richards 1991: 38) in constructing their own meanings for themselves while participating in mathematical discussions or problem-solving activities. In doing so, teachers seek to find ways to extend and expand the CD within which their students think and operate. This is achieved if students are willing to *interpret* the mathematical re-presentations of the teacher and internalize or rebut their teachers' mathematics and re-present their own mathematics (MOS).

« 8 » "Mathematics" is thus something that is interpreted and this, we believe, makes it something that "exists." George Berkeley's famous Latin-English coined dictum, "*esse is percipi*" (to exist is to be perceived) warrants the claim of "existence" of entities by virtue of their being perceptible (Berkeley 1949: part 1, §3). Not unlike Berkeley, we claim the existence of "mathematics" in the CD of mathematically communicating individuals to be warranted by its *interpretive* property because interpretation is a key aspect of perception. For perception to occur, minds need to interpret what eyes, ears, and hands see, hear, and feel.

Mathematics-to-Learner Negotiation

« 9 » In their effort to place "mathematics" at the disposal of their students, teachers need to make decisions as to what part of that mathematics is fit for their students (MFS) and how to go about interacting with students. This requires an anticipation of the teacher-learner interaction where teachers ask questions like

- What would the students think of the mathematical problem?
- What are the possible experiences of the students that I may tap into in order to help them make sense of what I am re-presenting?
- What possible difficulties may the students have when dealing with this concept?

« 10 » In order to start answering these questions, teachers need to concern themselves with how their students come to know (cf. **McCloughlin**'s Q1) and rely on models they build of the current or "similar" students' ways of coming to know. Teachers use this knowledge to choose how to interact with the students and we gave a number of examples of how this could be done (target article, §5). **Jaworski** (Q1) asked us to elaborate on how teachers' constructivist beliefs would be reflected in the way they approach this task. If teachers assume an RC stance, they need to see how their beliefs about knowledge and cognition are reflected in their thoughts and actions. Their belief that knowledge is actively constructed, not transmitted and received, should keep RC teachers from trying to force their methods onto the students. Rather, they seek to *learn* about students' reasoning and identify pathways between that reasoning and

MFS. The negotiation of those pathways is the orientation of students' ways of thinking. This comes with RC teachers' belief that knowledge serves students to make sense of their experiential worlds rather than to gain access to an absolute reality. Rather than presenting "mathematics" as a collection of "truths" about the universe, RC teachers will present it as a creation which can help humans explain and possibly predict experiential phenomena. They will try to motivate their learners to enrich their mathematics (MOS) by challenging them with situations where the students would feel the need to build or restructure their knowledge, in the pursuit of a more viable idea or way of reasoning. Furthermore, RC teachers hold that "knowing" is a state that can only be achieved when learners actively construct ideas in a process we generally refer to as "learning." We agree with Steffe (§10) that RC is a theory of knowing but, being such, it is necessarily a theory of learning. In fact, Glasersfeld's (1990a) first RC principle is more about the process of coming to know (learning) than it is about the nature or purpose of knowledge itself.

« 11 » RC teachers, therefore, make it their business to inquire about their students' current knowledge, in our case, MOS. Sometimes this is done "by posing problems, asking questions, [and] discussing" (Richards §6), as the teacher in our research was doing in most of the protocols. Perhaps this is why Steffe (§3) regards the protocols as "couched almost exclusively on the teacher's side of the interaction." Quite the contrary, we argue that Protocols 1–4 show the teacher asking questions in order to give students the opportunity to express themselves in the discussion so that he could build models of their MOS. In particular, Protocol 4 shows how Dan was given the opportunity to re-present, through verbal and bodily expressions, an analogy between an equality and a barbell with weights. As an example of student interaction (Steffe Q1), we present the contributions made by the students in all the protocols, including Protocols 5–6, where the teacher was observed not to engage in what we define as CT. The students interacted with the teacher by *interpreting* the teacher's expressions (utterances, gestures, drawings, demonstrations, etc.) and *acting* on those interpretations by re-presenting their own mathematical conjectures. Apart from expressions that one would expect in classroom discussions (re-presenting conceptual structures through verbal statements and questions, tone of voice, facial expressions, gestures, role-playing, etc.) the students had the opportunity to make re-presentations through drawings, symbol-writing, and activities on the interactive board, on their computers, and on paper. This is discussed in more detail in Borg & Hewitt (2015). Since our article was concerned with the analysis of the teacher's actions in the classroom with the help of the M-N-L framework we did not discuss student-student interactions and student-computer interactions unless the teacher was taking part in those interactions. This might have falsely depicted a picture of a classroom where the teacher did "not relinquish control [and] seemed to be centrally involved" (Steffe §6). Actually, the second half of each lesson was dedicated to letting students work in pairs on their computers on goal-oriented tasks and the teacher was only involved if students asked for assistance or if he *sensed* that his assistance might be required, as was the case in Protocol 1. This might address Steffe's concern expressed in Q2.

« 12 » Furthermore, to reply to Gash's Q2, the use of M-N-L to analyze CT is only a part of a larger research project that the first author is currently undertaking[2] in which part of the video analysis does concentrate on students' learning and interactions, particularly on whether and how students participated in cooperative learning, especially when they were working on their own in pairs. Nevertheless, what happened on the "Learner" end of M-N-L is pertinent to the overall question of CT and we are going to clarify some issues about this in the next section.

2| This is a PhD research project under the supervision of the second and third authors. In response to Gash's Q4, the lesson video analysis in this research project is indeed serving as a means of professional development for the first author with respect to his attempts to implement RC in his daily practices as a teacher, where his co-authors are indispensable mentors in giving meaning to the video observations.

Learner

« 13 » One of the most important features of M-N-L is that it characterizes CT as the type of teaching stance in which teachers learn from their students. This is done by encouraging students to *reflect* on a learning offer and contribute to class discussions or goal-oriented activities (in our case, these were set with the help of the software Grid Algebra). Tillema's Q2 refers to an important issue: how can constructivist (mathematics) teachers act on the belief that the meanings inferred in classroom communications do not have an ontological status outside of the ones that the communicating individuals confer on them? Teachers acknowledging this constructivist notion and who are aware that part of their duty as teachers is to negotiate their "mathematics" to the learners, need to listen to and immerse themselves in these communications and, if necessary, *orient* their students' thinking by presenting them with a "situation in which the students' network of explanatory concepts clearly turns out to be unsatisfactory" (Glasersfeld 2001: 10). This requires care that the teacher is not "merely 'tricking' the student to arrive at the same conception as the teacher" (McCloughlin §10). Teachers may contribute with their own meanings in the discussion and negotiate pathways between the "mathematics" in the syllabus and that of the students by creating "perturbations" in the minds of the students, the settlement of which is bound to result in students learning new concepts or modifying old ones. In addition, constructivist teachers must be open to letting their pedagogical and subject-matter knowledge be "perturbed" and to learn from the meanings that their students assign to the concepts being discussed.

« 14 » In order for teachers to create a CD in the classroom community (which includes themselves) they must seek to establish a state of taken-to-be-shared meanings. This means that when a student makes a significant contribution to the classroom interactions affecting the CD, the teacher either challenges or amplifies the contribution with the help of that student and by encouraging other students to join in the interaction. We could not include extensive protocols in the target article but to address Steffe's Q3, an individual student's contribution to classroom discussions and activities was indeed imputed to other members of the classroom community, but only after further discussion enabled the teacher to create models of students' ways of thinking in which students were observed to share the reasoning behind that contribution. In the case where students disagreed, the teacher challenged and encouraged students to reflect and discuss further, aiming to bring out the viability and unviability of the different perspectives and to help students adopt the explanation that was proved viable in that particular situation.

« 15 » Tillema (Q4) asks about distinguishing between a situation where students achieve a state of taken-to-be-shared meanings and agreed-upon concepts without making fundamental changes in their conceptual structures and a situation where such a state is achieved through the accommodations of new ideas in students' mental schemas. The distinction was not addressed explicitly in our analysis of the protocols, mostly due to our focus on the teacher's actions vis-à-vis CT. In Protocol 3, for instance, the students seemed to accept Dan's suggestion that the series 1, 2, 3,… could be seen as the one-times table. Given that they were all supposedly familiar with multiplication tables, we do not think that students had fundamentally changed their mathematics. Still, the possibility that they became aware of yet another way of looking at the positive integer series was an important foundation for what the teacher intended to introduce afterwards (the other multiplication tables and the relationship between numbers in different tables). There are other instances in the study, which we do not have the space to delve into here, where classroom interactions appeared to lead to deep and fundamental changes in the way students thought about some mathematical conventions. One such episode happened when students were presented with the statement $3 = 2 + 1$, where some students were adamant it was written in the wrong way since they were accustomed to seeing the equality symbol written at the end of a calculation and preceding the answer. It took more than one lesson episode to achieve a CD about the balancing property of the equality symbol, one of them being the episode from which Protocol 4 was taken.

Learner-to-Mathematics Negotiation

« 16 » What constructivist mathematics teachers do with what they observe and perceive regarding students' dealings with the classroom experiences is the second negotiation "road" between learner and "mathematics." Teachers build second-order experiential models of MOS that they use to revisit MFS. Our definition of MOS is narrower than that of Steffe (e.g., 1991a), which encompasses any student construction "that could be thought of as mathematical simply because they are human beings" (Steffe §8). As used in the M-N-L framework, MOS is the "mathematics" as described in §§9–11 above. Nevertheless, as Jerome Proulx (§1) states, MOS lies in the eyes of observers (teachers or researchers) who interpret what they observe according to their own experiences and the mathematical constructs they have construed for themselves.

« 17 » Jaworski (§4) associates our notion of teachers building second-order models of MOS, to Aaron Cicourel's concept of observing a social relation. We do not completely agree with this association. Cicourel (1973) was building on Alfred Schütz (1964), who distinguishes between someone observing interactions between actors in a *We-relation* and someone observing interactions between actors in a *They-relation*. In the former, the observer observes an interaction in which she is taking part. In the latter, the observer is not an actor in the interaction. Writing from the perspective of an observer, Schütz (1964: 55) says that "whereas my experience of a fellow-man [sic] in the We-relation is continuously modified and enriched by the experiences shared by us, this is not the case in the They-relation." Cicourel (1973), as cited by Jaworski (1994), is talking about observing a They-relation. In our article the teacher was involved in observing both types of relations. When acting as a teacher he observed a We-relation when he was involved in an interaction with the students and he also observed a They-relation when he was perceiving an interaction going on between the students in which he was not involved. When he was acting as a researcher, analyzing video recordings of lessons and computer activities, he was always observing a They-relation, even when he was involved in an interaction, because he could not somehow repeat the exact train of thoughts that went through his mind during the face-to-face encounter with the students.

« 18 » Schütz distinguishes a *pure* We-relation from a *concrete* We-relation. In the first, observers merely acknowledge that someone is "in front" of them but do not know if or how that person is comprehending them. In the latter, observers know how that someone is oriented toward them. To give an example, teachers may be aware that the students before them may or may not be paying attention to them (pure We-relation) but they can gain information about whether and how the students are comprehending them through "concrete manifestations of [the students'] subjective experiences" (Schütz 1964: 27) (concrete We-relation). Analyzing from an RC stance, the first author was not assuming the certainty Schütz (1964) seems to propose when claiming that in a concrete We-relation the observer could gain knowledge about how the other person is comprehending him/her. Whenever the teacher constructed models of MOS, both as a teacher and as a researcher, these could only be second-order models. Jaworski (Q2) asks how this position aligns with constructivism. It is precisely the assumption of uncertainty about students' conceptual structures that is key to the application of RC in the M-N-L framework. By talking about a second-order model formation of MOS (target article §5), we are implying that:

- The meaning students give to teacher-student interactions are subjective, individualistic, and unique. They might share a mathematical CD between them and with the teacher but that does not make their mathematical interpretations and re-presentations less unique.
- Students cannot share their understandings with the teacher (or with one another). As Glasersfeld (1995: 48) says, "the expression 'shared meaning' is misleading" because people can only talk about or make external re-presentations of those meanings. Teachers can only build a model of what their students are saying or manifesting by referring to associations of their own experiences with those words or actions.

« 19 » The RC stance of the M-N-L framework does not stop with teachers' constructions of models of MOS. Constructivist teachers allow their "mathematics" to be affected and often perturbed by what they learn about MOS (target article, §5f). The settlement of this perturbation results in a reviewed and sometimes modified MFS. In Q3, **Tillema** asks about the intended MFS and how this relates to the teacher's model of MOS. This is a difficult question to answer in a few words because there were quite a number of mathematical meanings that the teacher sought to discuss with the students during the course of the twenty double lessons spanning the whole scholastic year. Broadly speaking, the aim of the lessons was to help students familiarize themselves with formal algebraic notation with the help of Grid Algebra. This involved concepts like unknowns, variables, equalities, and order of operations. Since Grid Algebra uses the multiplication grid as a basis for its users to construct meanings about these concepts, students needed to get acquainted with the way it worked, particularly how specific numbers were designated to specific cells and how numbers or expressions in those cells related to others. In each lesson the teacher was repeatedly toing and froing between learner and "mathematics" by detecting or creating links between MOS and MFS. The few instances where the teacher was observed not to take MOS into consideration in his communication of MFS was taken as a failure on the part of the teacher to engage in CT, at least for that episode where the teacher seemed to disregard his sensitivity to students' individual constructions of MOS. We agree with **Gash**'s Q3 that teachers need to have the flexibility to postpone their actions on a student's contribution to the classroom discourse. His suggestion that when they need to do so teachers should acknowledge that student's intention to interact and show him/her that they would follow it up later is both pragmatic and constructivist. If teachers do this, they would still be showing their sensitivity to what students might be constructing and their intention to act on that sensitivity.

« 20 » The use of "negotiation" or "negotiated meanings" in constructivist literature is not necessarily used in the way we have described above and in the target article. **Thomas McCloughlin** (Q2) asked why we did not refer to the CAME project. Mundher Adhami, David Johnson & Michael Shayer's (1995) use of "negotiation" was derived from the work of Jörg Voigt (1994), whose concept of "negotiation" was the settlement of academic "conflicts" between teachers and students arising principally from their different points of view. Our use of "negotiation" is somewhat different. In M-N-L, teachers negotiate paths between what lies in the curriculum and what is already constructed in the learners' minds. Sometimes this is done by simply associating or assimilating MOS with MFS. At other times teachers need to adapt MFS while keeping in mind that they can never abandon the "mathematics" in the curriculum that they are duty-bound to help their students learn.

Abandoning the constructivist stance?

« 21 » **Jaworski** (Q4) questioned what would be lost in M-N-L if it abandoned constructivism as an overarching frame. Since M-N-L was devised from an RC stance we will attempt to answer this question with the assumption that RC was absent from this framework. If this were the case, the meaning of "mathematics" would be very different. It would simply be a body of knowledge that exists in its own right, irrespective of whether it was interpreted or re-presented. Teachers would not need to care about the individual experiences and hence diverse interpretations of the students. They would not need to learn anything about students' knowledge construction or, in our case, MOS. Their MFS would simply be the next topic in the syllabus and if students do not seem to be making sense of it they would assign more drill work until somehow the students started getting the correct answers. Teachers would not have to bother about understanding as long as students "performed" well. The aim of teaching would not be to help students construct meanings but for them to perform competently. Glasersfeld (1995) says that this is training rather than teaching.

« 22 » If RC were taken out of the picture, the whole rationale behind teachers' having to think of ways to negotiate between mathematics and learner would vanish. Why would teachers need to anticipate didactic processes according to models they create of experiences of the current or similar students (target article, §28) if mathematical concepts exist independently of experience? Why would teachers need to learn about their students, specifically about their construction of MOS if this had no bearing on what "mathematics" were on offer? The two-way negotiation road where teachers are also learners and learners are also teachers would be replaced by a conveyor-belt system that transported a body of a priori knowledge from teachers' mental databases to the students. Consequently, there would not be any reason for teachers to encourage reflection on the part of the learners, if all learners had to do were to accept what the knowledge conveyor belt brought their way. Teachers would not be the intermediary agents between "mathematics" and learners. Their role would simply be to place mathematical facts and techniques on the teacher side of the knowledge conveyor belt. The "negotiation" in M-*N*-L would not be a negotiation at all.

« 23 » We hope this re-affirms the indispensable constructivist stance of the M-N-L framework as much as we hope that this framework serves teachers and researchers to bring RC theory to school teaching practice.

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