& Lachance 2000). A mechanism showing the process of designing those tasks (e.g., nature, sequence) that makes explicit the choices of context, the iterative cycles of design, if any, and the conjectures the authors had about students' progression of thinking while working with those tasks could provide a stronger framework for studying the "bridging" process.

« 5 » Simon's (2013) design approach to learning through activity may offer a guide to structure this bridging framework through a sequence of four steps:

1 | Assess students' relevant mathematical conceptions;
2 | Articulate a learning goal;
3 | Specify an activity that students currently have available that can be the basis for developing the abstraction specified by the learning goal; and
4 | Design a task sequence and postulate a related learning process.

Geraniou and Mavrikis constructed a model of students' thinking in eXpresser, clearly described the two AWOT they have as learning goals, developed a sequence of tasks for reaching those goals and began their task design by having students' activity with eXpresser as the basis. What needs further investigation is the hypothetical learning process (Simon 2013, 2014), which takes the form of conjectures about student thinking and how the specific engineering of the task design and sequence may assist students in developing their knowledge and reach the AWOT goals. Questions that may guide this process include:

- What schemes and operations of AWOT were provoked in the initial context of eXpresser?
- How can similar schemes and operations be provoked in the new contexts?
- What could be the thinking of the student in those tasks that would explain "bridging"?

« 6 » Subsequently, the "bridging" process can be described by constructing models of how students' thinking developed through the research process (Cobb & Steffe 1983; Thompson 1982). These models will portray a trajectory of students' development of AWOT that consists of an explanation of students' initial schemes, explanations of changes in those schemes, and analysis of the contribution of the activities involved in those changes (Steffe 2003, 2004). A description of students' *intermediate changes of thinking* from the initial to the final AWOT would show the dynamic perspective of "bridging" as a process that evolves through design. The authors provide an example in their discussion of the development of the second AWOT, where they present the "intermediate step" of students' use of the eXpresser language to represent variables in the rule before they express their derived rules in a formal algebraic expression. "Bridging" would then be described as the process of how students' knowledge has been developed, modified, adapted or even refined during the learning process by identifying those "intermediate steps" as landmarks that build up to algebraic generalization.

« 7 » In this commentary, I have tried to contribute to the conversation by raising some issues that I consider essential to the "bridging" design and also presenting some suggestions of how students' thinking during the bridging process can be described and studied. My goal was to initiate a conversation of how a mechanism that explains the relationship between task design and students' development of knowledge can provide a framework for "bridging."

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## Building Bridges that are Functional and Structural

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**> Upshot** • In their article, Geraniou and Mavrikis describe an environment to help children explore algebraic relationships through pattern building. They report on transfer of learning from the computer to paper, but also implicit is transfer from concrete to abstract contexts. I make the case that transfer from abstract to concrete contexts should complement such approaches.

« 1 » In their target article, Eirini Geraniou and Manolis Mavrikis investigated how knowledge developed in a microworld environment, called MiGen, might transfer outside of that environment. They describe a sequence of "bridging" activities to aid students' transition from the computer to paper-based tasks. Students start with learning about the environment and constructing and describing generalised patterns within it, and then move on to paper-based activities that at first resemble the MiGen environment before taking the form of "textbook or exam-like tasks" (Figure 2). That is, the digital environment provides scaffolding to help students construct knowledge and the bridging activities provide fading to where "attention is purely on the mathematical notation and the mathematics of solving equations" (Hewitt 2014: 26).

« 2 » The MiGen environment needs to be learned and experienced for a sustained time. The authors report that students received two lessons designed to familiarise them with the environment, and conclude that students need "a long period of practice […] before transfer to mathematics can be deemed possible" (§26). Moreover, Geraniou and Mavrikis state that the literature and their own experiences "suggest that students rarely use ideas, concepts or strategies they seem to have acquired through their interactions with digital technologies in their mathematics classrooms" (§8). Microworlds take a lot of work, and success, in terms of transfer to non-digital contexts, is far from guaranteed. So is it worth the trouble?

« 3 » Constructionists argue that microworlds provide a powerful resource for immersing learners in mathematics. Abstract objects and concepts become tangible, allowing trial and error experimentation, mental reflection and discussion (Papert 1980). Students might then discover and explore ideas that are otherwise be inaccessible to them, and can be challenged in ways not always supported by typical classroom activities. Some readers of this journal will have experienced and studied this enabling power of microworlds. In my own research, students working in the SumPuzzles environment interacted with formal arithmetic equations in distinctly algebraic ways, focussing on structure not calculation, and did so with minimal explicit instruction (Jones & Pratt 2012). However, when the plug is pulled, is the knowledge constructed by the student switched off along with the computer? Work such as that by Geraniou and Mavrikis is important for exploring how students might be bridged to working with formal mathematics on paper, and helping to evaluate whether the scaffolding and fading payoff is worthwhile.

« 4 » Another form of transfer, or perhaps more accurately transition, is implied in the research; namely, the shift from arithmetic to algebraic ways of thinking. The authors report that many students were successful with the final bridging task, and so claim that students "can generalise and adopt [algebraic ways of thinking] when solving paper and pencil figural pattern generalisation tasks" (§26). However, there were exceptions in which students "reverted to their past experiences and worked out the answers for each consecutive term in a sequence" (§24). Researchers working in the early algebra field will be unsurprised by this. Years of learning arithmetic using conventional notation has been shown to develop "operational patterns" (McNeil & Alibali 2005), such as the expectation of a numeric answer and a propensity to perform calculations even when they are irrelevant to the task goal. Moreover, operational patterns are stubborn and can be triggered unhelpfully by traditional paper-based tasks (McNeil 2008). Carefully designed microworlds can free students from operational patterns in order to explore algebraic ways of thinking, but operations are likely to be prioritised again for some students when returning to more traditional presentations of mathematical tasks.

« 5 » At the heart of the MiGen philosophy is another important aspect of transfer, the shift from concrete to abstract knowledge. This has been a contentious issue of late, with a high-profile paper by Jennifer Kaminski, Vladimir Sloutsky and Andrew Heckler (2008) claiming mathematical ideas should be introduced in abstract contexts to ensure better transfer, and others challenging their finding (e.g.. De Bock et al. 2011). The use of generalised patterns to support algebraic ways of thinking has been termed "functional approaches" (Kirshner 2001). Appeals are made to children's experiences of pattern and regularity, and tasks are designed such that formal algebra offers a powerful medium for describing and generalising patterns. Alternatives, which are perhaps less visible in the literature, are "structural approaches." These start with the abstract (that is, formal symbols and their structural relationships, with no concern for real-world referents) and seek to nurture conceptual understanding that can be transferred to new contexts, be they abstract or concrete. Structural approaches perhaps have a tarnished reputation, sometimes being associated with "meaningless" arithmetic and algebraic drill. However, carefully designed tasks can enable interactions with formal notation and associated transformation rules in a rich, meaningful and educationally valuable way (Dörfler 2006). Microworlds that take this approach have been found to motivate engagement with algebraic ways of thinking about formal notation systems (Hewitt 2014; Jones & Pratt 2012).

« 6 » There are two potential reasons to consider structural approaches as complements to functional approaches. First, whereas functional approaches typically end with the production of a formal expression or equation used to describe a concrete referent (typically a pattern), structural approaches enable the exploration of how formal expressions can be transformed; the notation becomes a medium for *doing* mathematics rather than *describing* mathematics. Second, structural microworlds start with formal notation, a virtual and manipulable symbol system that closely resembles that typically seen in textbooks and classrooms. Therefore, transfer from a digital to a paper-based domain might be relatively natural and intuitive for many students.

« 7 » We can assume that constructionist approaches to introducing formal algebra naturally align with both functional and structural approaches. Indeed, both approaches have been shown to lend themselves to the design of microworlds that enable tangible exploration and testing of conjectures such that formal symbol systems become a natural and useful medium of mathematical learning. Ideally, we might want learners to shift flexibly between thinking about concrete referents such as generalisable patterns, and thinking with formal symbols and their transformation rules. Such a fluid and dialectic mixed-approach might be expected to strengthen algebraic experience and understanding, and so promote transfer in the broadest sense of the term.

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